

Relational Contracts, Limited Liability, and Employment Dynamics*

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Abstract

This paper studies a model of relational contracts of imperfect public monitoring when the agent has limited liability. The optimal relational contract provides joint and definitive implications on worker's job security, average earning, and the sensitivity of pay to performance over time as the employment relationship progresses. In addition, we analyze how the employment dynamics change with respect to the surplus in the relationship and firm's ability to commit to long-term long contract. Our results shed light on empirical findings that relate employment dynamics with firm characteristics.

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1 Introduction

Employment relationships have two salient features. First, the success of a relationship depends on proper provision of incentives. In particular, a worker has a moral hazard problem when exerting effort is costly and the level of effort is imperfectly observed by the firm. A prominent line of research has emphasized the importance and consequences of moral hazard within firms; see Gibbons and Waldman (1999), Hart and Holmstrom (1987), Prendergast (1999) for reviews. Second, the employment relationships are typically ongoing. Hall (1982) reports that a quarter of the workforce in the U.S. has jobs that last more than 20 years. This repeated aspect of the relationship implies that current actions taken by the worker and the firm have consequences in the future.

This paper studies the combined effect of “moral hazard” and “repeated interaction” on employment dynamics. Specifically, we ask, how should a firm dynamically structure its incentive system to induce effort from the worker? What implications does the optimal dynamic incentive structure have on the worker’s turnover and compensation dynamics? Since the seminal paper of Lazear (1979), these important questions have been studied extensively.¹ This paper contributes to the literature by showing that dynamic moral hazard can provide joint and definitive implications on worker’s job security, average earning, and the sensitivity of pay to performance over time as the employment relationship progresses. Equally important, our paper asks, how the optimal incentive structure and the associated employment dynamics change with respect to the surplus in the relationship and the firm’s ability to commit to long-term contracts? Our answers to this question shed light on the empirical literature that relates employment patterns with firm characteristics.²

We analyze a model of relational contracts with imperfect public monitoring when the agent has limited liability and compare the optimal relational contract with the optimal long-term contract. Without limited liability, our model is a special case of Levin (2003) with binary effort levels and outcomes. The limited liability constraint is modelled such that the agent’s compensation in each period cannot fall below an exogenously given wage floor. The limited liability assumption is a convenient and commonly used modeling device in moral hazard models. It helps simplify the analysis by abstracting away issues from intertemporal insurance.³ Just as in Levin (2003), we define each relational contract as a Perfect Public Equilibrium (PPE) of the model and the optimal relational contract as the PPE that maximizes the principal’s payoff.

¹Earlier theoretical models on this topic include Rogerson (1982), Spear and Srivastava (1987). The efficiency wage literature a la Shapiro and Stiglitz (1984) has focused on the general-equilibrium implications of repeated moral hazard. For a discussion on the empirical implications of dynamic moral hazard models, see Hutchens (1989).

²See Oi and Idson (1999) for a survey.

³Innes (1990) and Kadan and Swinkels (2008) provide theoretical analysis of moral hazard with limited liability in the static setting.

We completely characterize the set of PPE payoffs and the resulting optimal relational contract using the method developed by Abreu, Pearce, Stacchetti (1990).

Our results show that limited liability drastically changes the nature and structure of the optimal relational contract. Without limited liability, Levin (2003) implies that the optimal relational contract in our setting is efficient, stationary, and the firm can extract all of the surplus in the relationship. When limited liability is present, some rent must be given to the worker to induce effort. In designing the optimal dynamic incentive structure, the firm faces the trade-off of efficiency and rent extraction. The optimal relational contract is no longer efficient nor is it stationary. But the structure of the optimal relational contract is surprisingly simple.

Let us first describe our benchmark optimal relational contract where the surplus of the relationship is high and the limited liability is sufficiently binding: i.e. both the firm and the worker have lots of rent in the relationship. In this case, the employment relationship begins with a “probation phase,” during which the worker receives a constant wage equaling the wage floor. Depending on his performances, the worker is either terminated or transitions to the “tenure phase” in which he is given permanent employment. In the tenure phase, the worker receives the same incentive contract every period: he receives a bonus whenever his performance is successful.

Probation phase is widely used across jobs and is an important feature for many professional jobs. Existing models on probation phase have typically focused its role in selecting the workers; see for example Bull and Tedeschi (1989), Sadanand et al. (1989), Weiss and Wang (1990), and Wang and Weiss (1998). Here, we show that the probation phase can also arise as part of the optimal dynamic incentive device as a method for the firm to extract rent from the worker. Interestingly, in perhaps the most famous example of designing compensation policy to solve moral hazard problem, Henry Ford’s five-dollar-day program specifies a six-month probationary period. During these six months, the worker is paid the prevailing market wage of about 2.5 dollar per day before they can qualify for 5 dollar per day.⁴

In general, the duration of the probationary period predicted by our model is stochastic. But in some cases, there can be a fixed date on which the worker will be terminated unless he receives tenure on or before that date. This happens, for example, when successful performance is unlikely. Such turnover process is similar to “up-or-out” contracts that feature prominently in professional jobs. An interesting feature of “up-or-out” is that its turnover rate is (degenerate) inverse-U-shaped with respect to time on the job. That turnover rates may initially increase

⁴This program also reflects the relational aspect of employment. In Henry Ford’s Five-Dollar-A-Day program, there has been fear among workers that Ford may systematically fire workers before their six-month probationary period running out. Raff and Summers (1989) speculate that “It is plausible that such fear constrains Ford from imposing too long a probationary period, whatever his impulse about the appropriate length may have been.”

is a celebrated prediction of learning models (Jovanovic (1979)) and has been documented in empirical work (Farber (1994)). Our model shows that this can also be the consequence of optimal relational contract in which the firm minimizes the efficiency loss while extracting rent from the worker. In particular, the firm extracts rent from the worker by using termination as an incentive device to save for bonus payment. But termination destroys surplus, so the firm postpones it to mitigate efficiency loss. This reduces the turnover rate at the beginning of the relationship.

The optimal relational contract also has implications on the compensation dynamics. First, the compensation is deferred in the sense that the worker's average pay increases upon receiving tenure. The deferred compensation is an important prediction of moral hazard models; see Lazear (1979) for seminal work on this topic and Hutchens (1989) for a survey. As we will explain in Section 4, the reason for deferred compensation is different from earlier moral hazard models, and the pattern of deferred compensation is also different. In particular, at an individual level, our model implies deferred compensation across employment phases but not within. At an aggregate level, our model predicts that the average earnings growth eventually goes to zero.

Second, the model predicts that compensation becomes more sensitive to performance over time. Under the optimal relational contract, the worker's pay is insensitive to performance in the earlier stage of the employment and performance bonus is given out only after the worker has passed the probation phase. At an aggregate level, the average sensitivity to performance increases over time as more workers passed the probation phase and become eligible for performance bonus. The increased sensitivity of pay to performance has been found in some segment of the labor market in the U.S.: see Gibbons and Murphy (1992) for CEOs, Gompers and Lerner (1999) for venture capitalists; and Misra, Coughlan and Narasimhan (2005) for salespeople. Hashimoto (1979) reports that the bonus to wage ratio is increasing with experience on the job in Japanese firms.

The simplicity of the employment structure reflects a key principle in designing dynamic incentive systems for jobs with rents: job security (as part of the reward) should be *frontloaded* as much as possible. In particular, when the job has rent, being able to work for the firm is valuable for the worker. In this case, the firm can choose between two types of rewards: giving bonus payment or giving future job security. Bonus is simply a transfer between the two parties. On the other hand, giving out future job security increases the chance the relationship is going to last, and the firm can profit from the increase in future surplus. In other words, rewarding with job security increases the size of the pie yet paying bonus simply divides it. It is to the interest of the firm not to miss any chance of increasing the size of the pie: the worker should be rewarded with job security in the early stage of the employment relationship.

In our benchmark case of high surplus, the principal is able to completely frontload job

security as reward in the sense that no bonus is paid out until the worker receives tenure, after which the worker is motivated by bonus payment alone. In our low-surplus case, the principal cannot rely solely on bonus induce effort: the bonus required will exceed the surplus in the relationship and the principal will renege in giving it out. This means the principal always has to rely on the threat of termination to induce effort and thus is unable to award the agent with tenure. In this case, the optimal relational contract results in *incomplete frontloading* in the sense the principal frontloads only up to a point which falls short of tenure. The worker is again rewarded with job security in the beginning of the relationship. Only after he has earned high enough level of job security, the worker starts to receive a bonus for high output. And each low output jeopardizes his future job security. In the relationship, the principal always carries both carrot (bonus) and stick (less job security) to induce effort. But the carrot is given out only for workers with sufficient job security.

Comparing the optimal relational contract between high and low surplus, we show that high surplus firms provide more job security, higher average pay, steeper earning profile, and higher sensitivity to pay to performance. Some implications fit with the existing evidence linking firm characteristics with worker pay. For example, Abowd, Kramarz, and Margolis (1999) find that controlling for personal fixed effect, firms that pay higher wages are also more productive and profitable. For other implications, we do not know how well they fit the data. Given the increased availability of data matching workers and firms, we expect more empirical work linking firm profitability with employment dynamics to be conducted. Our results point to further empirical tests in this area.

We also examine the implications of firm's ability to commit to long-term contracts on employment dynamics. Our results show that commitment does not change the employment dynamics when the relationship has high surplus. In the low-surplus case, commitment helps the firm to better frontload job security as reward. In particular, the employment relationship again starts with a probation phase and the worker may receive permanent employment. Our results imply that when a relationship has low surplus, commitment leads to better job security, higher pay, steeper earning profile, and higher sensitivity of pay to performance. Moreover, these effects are larger for firms with low profitability.

To the extent that a firm's employee size may proxy for its ability to commit to long-term contracts, our results are consistent with many findings in the literature on firm size and employee pay. This literature finds that larger firms are associated with longer tenure (Idson (1996)); higher pay (Oi and Idson (1999)), steeper earning slope (Hashimoto and Raisian (1985)), and higher bonus to wage ratio (Hashimoto (1979)). On the other hand, the existing literature has not focused on the joint effect of size and profitability, and it will be interesting to test this empirically.

This paper contributes to three literatures. First, it adds to the growing literature that studies the dynamics of relational contracts. Classic models of relational contracts, such as Bull (1987), MacLeod and Malcolmson (1988), and (the first part of) Levin (2003), have focused on the conditions under which cooperation can be sustained, and the optimal relational contracts in these models are stationary. This new literature has examined the evolution of relationship when additional frictions are present: Halac (2008) studies a model in which the principal's type is private information. In Yang (2009), the agent's type is private information. Fuchs (2007) and (second part) of Levin (2003) both analyze the case in which the output is privately observed by the principal. In Chassang (forthcoming), the efficient production function is unknown, and the agent's moral hazard problem is linked to a problem of experimentation.

In our model, the source of friction is the limit liability of the agent, so (monetary) transfer between the principal and the agent is constrained. We show that this constraint leads to dynamics that has many implications in the labor market. A closely related paper that also generates dynamics in relational contracts in the presence of moral hazard and limited liability is Thomas and Worrall (2007). In their model, production has no uncertainty, so the relationship does not terminate, and their focus is on the dynamics of effort provision. A distinctive feature of their model is that effort may be overprovided in the earlier stage of employment.

While we have focused the implications of the model when the limited liability is sufficiently binding (high wage floor), our analysis also reveals an important source of inefficiency when the limited liability constraint is only mildly binding. In this case, the inefficiency involves suspension of production in the equilibrium play: the agent stays in the relationship but shirks. When production is suspended, the surplus in the relationship in the period is negative, and the agent's payoff falls below his minimal static Nash payoff (though that will never happen for the principal). A detailed study on similar destruction of surplus in a political economy setting can be found in Pedro i Miguel and Yared (2009). They study a similar repeated principal-agent model without commitment where frictionless transfer is impossible. In their model, the principal can take costly actions to punish the agent and they focus on the likelihood, duration, and intensity of punishment.

Second, by characterizing the optimal contract when the principal can commit, this paper also contributes to vibrant literature of optimal dynamic contract. This literature takes the optimal contracting view and studies its implications in diverse economic situations: see Biais et al (2007), Biais et al (forthcoming), DeMarzo and Fishman (2007), and DeMarzo and Sannikov (2006), and He (2009) for application in finance; Clementi and Hopenhayn (2006) in industrial organization and firm dynamics; Myerson (2008) in political economy; Lewis (2009) and Lewis and Ottaviani (2008) in search theory. Ours is an application in labor economics and focus on employment dynamics.

Finally, our paper also belongs to the labor literature that models employment dynamics. Since this is a vast literature and employment dynamics is naturally affected by many factors including but not restricted to human capital, learning, and search, we only relate our paper to the strand of literature that focuses on agency costs. In a series of seminal papers, Lazear (1979), (1981) and show that agency costs can account for important features of the labor market including up-sloping earning profile, hour restrictions, and mandatory retirement. Lazear and Moore (1982) argue that agency cost can be an important reason why the slopes of earning profiles are steeper for employed workers than for self-employed ones. Akerlof and Katz (1989) highlight the importance of giving rents to the worker in these models when the worker has limited liability. In the models above, the worker is finitely-lived and the firm can commit to long-term contracts. Yang (2009) shows that even if firms cannot commit to long-term contracts and the workers are infinitely-lived, the upward-sloping earning profile can arise when the workers are heterogeneous and they have private information about their productivities. Our paper adds to the literature by providing joint implications on earning and turnover dynamics. Moreover, we focus on how the employment dynamics are affected by the surplus in the relationship and the firm's ability to commit. On the other hand, given the worker is infinitely lived in this model, we cannot easily explain features such as mandatory retirement in our framework.

The rest of the paper is organized as follows. We set up the model in Section 2. The optimal relational contract is characterized in Section 3. We discuss the empirical implications of the optimal relational contract in Section 4. Section 5 concludes.

2 Setup

2.1 Environment

There is one principal and one agent. Both are risk neutral, infinitely lived, and have a common discount factor of δ . Time is discrete and indexed by $t \in \{1, 2, \dots, \infty\}$.

At the beginning of each period t , the principal decides whether to offer a contract to the agent: $d_t^P \in \{0, 1\}$. If no contract is offered ($d_t^P = 0$), the two parties receive their outside options in this period. The agent's per period outside option is \underline{w} ; the principal's per period outside option is \underline{v} . If the contract is offered, it specifies a legally enforceable wage w_t . We assume that

$$w_t \geq \underline{w},$$

where \underline{w} is an exogenously given wage floor.

In many relational contracts models, the contract also includes a discretionary bonus at the end of the period. In our setup, this discretionary bonus is postponed and becomes part the

wage offered at the beginning of next period. It can be shown that these two setups give rise to the same set of equilibrium payoffs.⁵ We choose our setup because it simplifies notations and facilitates our comparison to the efficiency wage models. Results in this model can be readily translated into a version with discretionary bonus.

If the principal offers the contract, the agent chooses $d_t^A \in \{0, 1\}$. If he rejects the contract ($d_t^A = 0$), the two parties receive their outside options. Otherwise, the relationship starts, and the principal pays out the wage w_t . The agent chooses effort $e_t \in \{0, 1\}$, and the output $Y_t \in \{0, y\}$ is realized. If the agent works ($e_t = 1$), the output Y_t is y with probability $p \in (0, 1)$ and 0 with probability $1 - p$. If the agent shirks ($e_t = 0$), the outcome is y with probability $q < p$. If he chooses to work, the agent incurs an effort cost of c . To make the analysis interesting, we assume that the value of the relationship exceeds the sum of outside options if and only the agent works:

$$py - c > \underline{u} + \underline{v} > qy.$$

The effort of the agent is his private information, the output is publicly observable, but not contractible. After the output is realized, both the principal and the agent observe at the end of the period $x_t \in [0, 1]$, drawn from a public randomization device.⁶ We summarize the timing by the graph below.

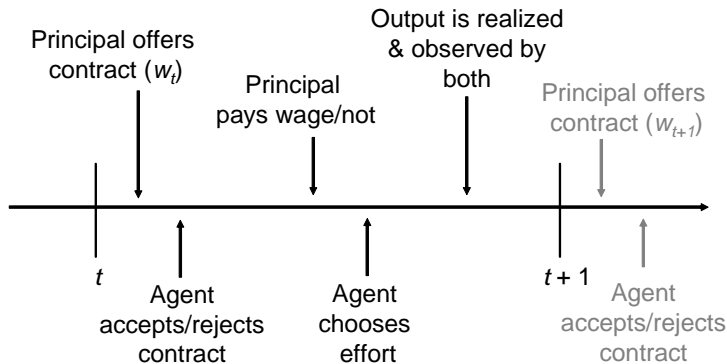


Figure 1: Timeline

At the beginning of any period t , the expected payoff to the principal and the agent are

⁵Malcomson and MacLeod (1998) provides a formal proof in the symmetric information case. Their proof can be adapted to the current case.

⁶The public randomization device is a commonly-made assumption in models of repeated games to convexify the equilibrium payoffs; see for example Mailath and Samuelson (2006), Section 3.4 for a discussion of its roles.

given by

$$\begin{aligned}
v_t &= (1 - \delta) \sum_{\tau=t} ((1 - d_t^P d_t^A) \underline{v} + d_t^P d_t^A (y(q + (p - q)e_t) - w_t)); \\
u_t &= (1 - \delta) \sum_{\tau=t} ((1 - d_t^P d_t^A) \underline{u} + d_t^P d_t^A (w_t - ce_t)),
\end{aligned}$$

where we multiplied through by $(1 - \delta)$ to express payoffs as per period averages.

Our environment is an infinitely repeated game with imperfect public monitoring, and we follow the literature to use public perfect equilibrium (PPE) as the solution concept.⁷ PPE is defined as strategy profiles such that a) players use strategies that depend on the public history. and b) the strategy profiles following any public history is a Nash Equilibrium.

Formally, we denote $h_t = \{d_t^P, w_t, d_t^A, y_t, x_t\}$ as the public events that happen in period t . Denote $h^t = \{h_n\}_{n=0}^{t-1}$ as a public history path at the beginning of period t , and $h^1 = \emptyset$. Let $H^t = \{h^t\}$ be the set of public history paths till time t , and define $H = \cup_t H^t$ as the set of public histories.

In period t , the action of the principal is to choose D_t^P from H^t to $\{0, 1\}$ and W_t from H^t to $[\underline{w}, \infty)$. The public strategy of the principal is $\{D_t^P, W_t\}_{t=1}^\infty$. The agent chooses D_t^A from $H^t \cup \{w_t\}$ to $\{0, 1\}$ and E_t from $H^t \cup \{w_t\}$ to $\{0, 1\}$. And the strategy of the agent is $\{D_t^A, E_t\}_{t=1}^\infty$. We allow the players to play mixed strategies and denote σ^P and σ^A to be the principal and the agent's mixed public strategies.

We denote $v(\sigma^P, \sigma^A | h_t)$ and $u(\sigma^P, \sigma^A | h_t)$ as the principal's and the agent's expected payoff following public history h_t . Our goal is to characterize the PPE that maximizes the principal's payoff at the beginning of period 1 ($v(\sigma^P, \sigma^A | h_1)$).

2.2 Comments

We describe in this subsection some key assumptions in our model: why we choose them and how they relate to the existing work in the literature. The three key assumptions are a) lack of long-term contract, b) limit liability, and c) imperfect public monitoring.

The lack of long-term contract and the repeated interactions between the principal and the agent makes this model "relational contracts with imperfect public monitoring." We think that relational contracts describe a large fraction of the labor market because most employment

⁷This model is a game of imperfect monitoring with "product structure", in the sense that the output depends on the agent's effort alone. It follows that our restriction to PPEs is without loss of generality; see Fudenberg and Levine (1994).

relationships are ongoing, and many last for a lifetime.⁸ While long-term contracts are desirable in repeated interactions, several factors may lead firms to leave many details of the employment relationship outside the contract. The cost of drafting a complicated contract is an important factor. Moreover, if the work is complex and its quality dimension is important, many aspects of the job cannot be precisely communicated to the court so as to be contracted on. Even in relatively simple jobs, long-term contracts may be infeasible when their enforcement become an issue: workers may fear that the court will be biased toward the firm.⁹

By modeling employment relationships as relational contracts, we hope to link characteristics of the firms, including their profitability and its size, with the employment dynamics within. We will show that relational contracts provide a natural setting to study how difference levels of surplus in the relationship can lead to different incentive structure and importantly affect its employment dynamics. In addition, we also solve the model assuming that the principal can write long-term contracts and compare the results. This comparison provides a rigorous foundation to earlier efficiency wage models (with bond-posting or deferred wage payments), which invoke reputation concerns of the firms to justify that firms will not cheat on the workers. Moreover, the comparison offers insights to how firm's ability to commit to long-term contracts affect employment dynamics.

The second key assumption, the limited liability constraint, is captured by the wage floor \underline{w} . There are many interpretations of \underline{w} ,¹⁰ and we do not tie it to any particular one. What is important for us is that \underline{w} represents the extent to which the principal can punish the agent and thus the amount of rent the worker has in the relationship. This restriction of punishment corresponds to the degree the worker can post bonds in efficiency wage models. When workers can post bonds (and forfeit them when caught shirking), it is no longer necessary for the firms to share rents with them, and efficiency wage models a la Shapiro and Stiglitz (1984) will collapse.¹¹ In a more general environment, Levin (2003) shows that the optimal relational contracts without limited liability can be stationary. Dickens et al (1990) provide many reasons for why this restriction is plausible, emphasizing the unwillingness of the court in enforcing contracts in which the penalty exceeds the damage.

The third key assumption is the imperfect public monitoring: the agent's effort is unobservable and the output is stochastic. Many earlier efficiency wage models also assume that effort is

⁸Hall (1982) finds that a quarter of the jobs in the U.S. lasts for more than twenty years. Recent estimates, including Ureta (1992), Farber (1999) and Stevens (2005), give similar results.

⁹Nevin (1954) reports fear of workers at Ford's plant that he will renege on the Five-Dollar Plan after the workers go through a six-month probation period.

¹⁰In some cases, \underline{w} can be thought of as minimum wage. In other examples, $\underline{w} = 0$ and represents the constraint that the principal cannot take money away from the agent. In another example, firms sometimes tie wages to jobs, and \underline{w} can be thought of as the lowest wage attached to the job. Prendergast (1993) shows that such commitment can help induce firm-specific human capital.

¹¹See Carmichael (1985) for a discussion.

worker's private information, one important difference here is that we have $p < 1$, so the output can be low even if the worker puts in effort. As such, the model does not apply well to examples in which Type II errors are unlikely, for example innocent workers caught stealing. Instead, it better describes examples in which the true cause of an undesirable outcome is not easily known. For example, if a worker calls in sick, it is not easy to tell whether the worker is truly sick or just lying. By allowing for Type II errors in the production, this model generates wage and turnover dynamics that are missing in earlier efficiency wage models.¹²

The production function in our model has been widely used in the literature, see for example Holmstrom and Tirole (1993) and Fuchs (2007). However, a model with binary actions are clearly restrictive and oversimplified.¹³ In particular, the binary action assumption rules out the dynamics of effort.¹⁴ We make these assumptions to make the model tractable. Models with multiple levels of effort are significantly more difficult to analyze.

3 Analysis

In this section, we characterize the optimal relational contract using the method developed by Abreu, Pearce, and Stacchetti (1990) (APS hereafter). APS shows how to construct the equilibrium strategies from the set of PPE payoffs and how this payoff set can be solved recursively. In Subsection 3.1, we characterize the PPE payoff set. In Subsection 3.2, we derive properties of the optimal relational contract from the PPE payoff set. With a few exceptions, formal derivations are contained in the appendix.

3.1 Characterizing the PPE Payoffs

To characterize the set of PPE payoffs, we first denote (u, v) as a payoff pair in which the agent receives u and the principal receives v . Suppose there is a PPE that gives the agent a payoff of u . Denote $f(u)$ as the principal's highest possible PPE payoff given the agent receives u . We show in the appendix that every (u, v) for $v \in [v, f(u)]$ can be supported as an equilibrium payoff, so we refer to f as the Pareto frontier of PPE payoffs. The goal of this subsection is to solve for f . But first, we review the benchmark case of relational contract without limited liability. The structure of f becomes more clear when compared to it.

¹²In particular, in earlier efficiency models, the worker will be fired immediately if caught shirking, yet he is never caught shirking and never fired in equilibrium.

¹³Given the agent is risk-neutral in this model, we feel that limiting the outcomes does not do too much damage. Levin (2003) shows that when the agent is risk neutral, the multiple-output model behaves like a two-output model. The total compensation the agent receives depends only on the "sign" of the likelihood ratio associated with the output.

¹⁴As long as the relationship persists, the agent always chooses $e = 1$. Thomas and Worall (2007) provides a rich dynamics of efforts when they are observable.

3.1.1 Benchmark: Relational Contract without Limited Liability

Without limited liability, our model becomes a special case of Levin (2003) with a small difference in timing that we discuss towards the end of the section. Levin (2003) implies that the optimal relational contract can be implemented by a sequence of short-term contracts that are stationary over time. Each contract specifies a base wage w_s and a bonus b_s if the output is successful. To induce effort, we need the (normalized) marginal benefit of effort exceed its cost:

$$(1 - \delta)(p - q)b_s \geq (1 - \delta)c, \quad (1)$$

where recall $p - q$ is the additional success probability through effort, so the left hand side is the marginal benefit of effort. For the relational contract to be sustainable, Levin (2003) shows that we need the principal's maximum reneging temptation to not exceed the surplus in the relationship:

$$(1 - \delta)b_s \leq \delta(py - c - \underline{u} - \underline{v}).$$

Combining these two inequalities, the necessary and sufficient condition for the existence of a relational contract with effort becomes:

$$(py - c) - (\underline{u} + \underline{v}) \geq k \equiv \frac{(1 - \delta)c}{\delta(p - q)}, \quad (2)$$

where k can be thought of as the minimum reward for high output to sustain effort.

When Eq(2) holds, the optimal relational contract can be found in a two-step procedure. First, choose the bonus b_s big enough to induce effort. In particular, we may set

$$b_s = \frac{c}{p - q},$$

so the worker is just willing to put in effort. Second, set the base wage w_s to extract the whole surplus from the relationship. In particular, the principal sets w_s so that the agent's payoff is exactly equal to his outside option:

$$w_s + pb_s - c = \underline{u}.$$

This gives that

$$w_s = \underline{u} - \frac{qc}{p - q}.$$

In this optimal relational contract, the agent receives \underline{u} , and the principal captures the whole surplus by having a payoff of $py - c - \underline{u}$.

Now note that in our setup, the principal does not give a bonus at the end of a period. Instead, the bonus is essentially paid out as part of the wage in the following period. This

allows for a direct translation of results in Levin (2003) to ours. In particular, the optimal "stationary" relational contract becomes

$$\begin{aligned} w_1 &= w_s \\ w_t &= w_s + 1_{\{Y_{t-1}=y\}} \left(\frac{b_s}{\delta} \right), \text{ for } t > 1 \end{aligned}$$

where noting that paying out b_s at the end of a period is equivalent paying out $\frac{b_s}{\delta}$ at the beginning of next period.

To characterize the Pareto frontier, we can use the transfer at the beginning of the first period to reallocate the payoffs between the principal and agent (and keep the rest of the strategies fixed), as long as the resulting payoffs are both greater than or equal to their outside options. More formally, if (u, v) is a PPE payoff, then all the payoffs in the set

$$\{(u', v') | u' + v' = u + v, u' \geq \underline{u}, v' \geq \underline{v}\}$$

are again PPE payoffs. This implies that as long as there is enough surplus in the relationship (Eq(2) holds), then Pareto frontier is the negative 45 degree line segment originating from the payoff pair given by the optimal relational contract $((\underline{u}, py - c - \underline{u}))$, as Figure 2 below illustrates.

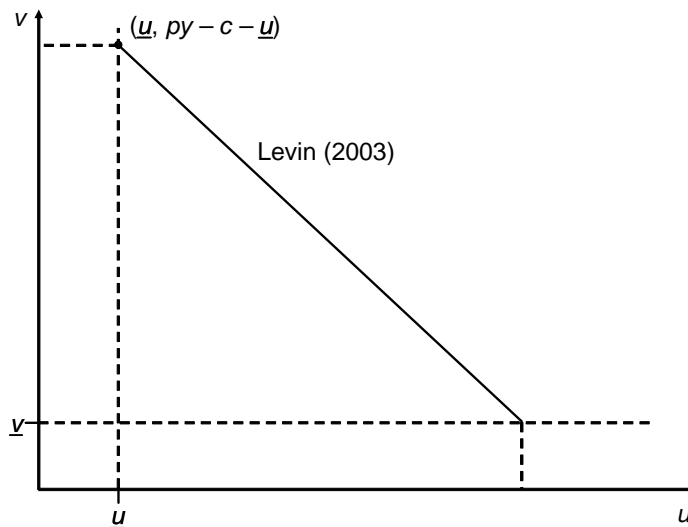


Figure 2: Pareto Frontier, w/o Limited Liability

Without limited liability, there is no trade-off between the efficiency of the relationship and the principal's payoff. The optimal relational contract can be found by first identifying the most

efficient relational contract and then setting a base wage (or making a first-period transfer) so that the principal captures the whole surplus. When limited liability is present, this is no longer possible and some rent will have to be given to the agent. The optimal relational contract will be inefficient and reflects how the principal optimally extracts rent from the relationship.

3.1.2 Pareto Frontier of Relational Contract with Limited Liability

When limited liability is present, the Pareto frontier f is no longer a line segment with slope -1 . Instead, it is cut into three regions. To the right of a high threshold (u_e), the Pareto frontier is again a straight line with a slope of -1 . This region corresponds to the Pareto frontier without limited liability. When the surplus in the relationship is high, this region is self-generating in the sense that every payoff pair can be implemented by a sequence of stationary contracts (with proper transfer in period 1). This region corresponds to the "tenure phase."

To the left of a low threshold (u_0), the Pareto frontier is line segment between $(\underline{u}, f(\underline{u}))$ and $(u_0, f(u_0))$. Every point in this region is sustained by a public randomization between the two end points. Inside this region, the agent's payoff is too low to induce effort, so inefficient actions take place with positive probability. In particular, if the wage floor \underline{w} is sufficiently high, then $f(\underline{u}) = \underline{v}$ and the inefficiency takes the form of termination. Otherwise, $f(\underline{u}) > \underline{v}$, and the worker will shirk and get paid \underline{w} in some period. Note that when the worker shirks, the relationship has a negative surplus (relative to the outside options). When there is limited liability, such play can never happen.

In the middle region, the Pareto frontier is described by the solution to a functional equation to be described shortly. In this region, the agent puts in effort and is paid \underline{w} . We denote this region the probation region. Figure 3 depicts a possible shape of the Pareto frontier of the PPE payoffs. It is drawn based on the assumption that the efficient outcome is sustainable and the limited liability is sufficiently binding. A formal theorem for the characterization of the Pareto frontier will be provided at the end of this subsection and its proof can be found in the appendix. We give a sketch of the proof below, and readers who are more interested in applications of the model can skip them.

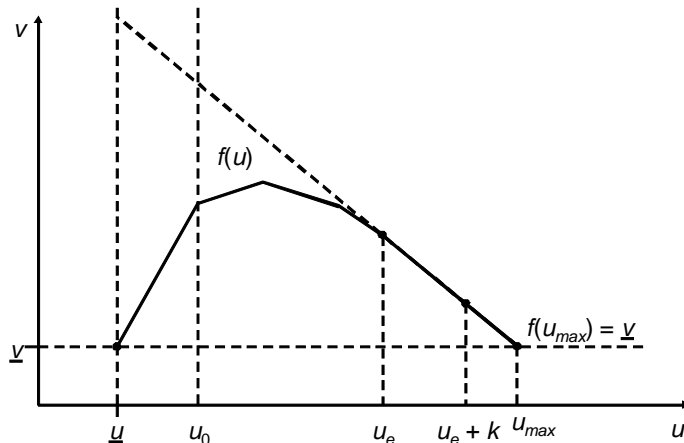


Figure 3: Possible Pareto Frontier, w/ Limited Liability

We define u_0 as the smallest u with which $(u, f(u))$ is obtained by requiring the agent to put in effort in period 1, and define u_e as the smallest PPE payoff of the agent that maximizes the sum of the equilibrium payoffs of the principal and agent. The main theorem of this subsection, Theorem 1, is formally proved through establishing Lemmas 1-7 and Corollary 1a in the appendix. In Lemma 1 we establish that to the right of u_e , $f(u)$ is characterized by the straight line $f(u) = f(u_e) + u_e - u$ with slope -1 . The argument is that if $(u_e, f(u_e))$ is an equilibrium payoff pair, then through increasing the first period transfer from the principal, any point on the straight line to the right of u_e is attainable. On the other hand, if any point strictly above the straight line is attainable, then it violates the definition of u_e . Next, based on concavity of f and the definition of u_e , it immediately follows that $f'(u) > -1$ for $u < u_e$. This point is summarized in Corollary 1a.

Next, we show in Lemma 2 that following any on-the-equilibrium-path history h in which $u_h \in (\underline{u}, u_e]$, $w_h = \underline{w}$ must hold. The first step is to show that $w_1 = \underline{w}$ must hold in period 1 if $u_1 \in (\underline{u}, u_e]$. The basic argument is that if $(u, f(u))$ is an equilibrium payoff pair and $w_1 > \underline{w}$, then the first period wage can be lowered by ε to achieve the payoff pair $(u - \varepsilon, f(u) + \varepsilon)$. This contradicts the fact that $f'(u) > -1$ for $u < u_e$. The proof is completed by noting that the principal's payoff following any history that happens with a positive probability should be $f(u_h)$. Since this result is true for any history, it immediately implies that no bonus will be paid out for good outcomes until the agent's payoff exceeds u_e .

Lemmas 3-5 focus on deriving the property of $f(u)$ for $u \in [\underline{u}, u_0]$. In Lemma 3, we establish

that for $u \in [\underline{u}, u_0]$, the Pareto frontier is characterized by the straight line

$$f(u) = f(\underline{u}) + \frac{u - \underline{u}}{u_0 - \underline{u}}(f(u_0) - f(\underline{u})),$$

for some $f(\underline{u}) \geq \underline{v}$. The basic idea is that when $u < u_0$, effort cannot be sustained by pure strategy. So the best the principal can do is to mix between $(\underline{u}, f(\underline{u}))$ and $(u_0, f(u_0))$. If the randomization gives the agent u_0 , which happens with probability $(u - \underline{u}) / (u_0 - \underline{u})$, then the agent will continue to exert effort. Otherwise, the agent receives \underline{u} and will not exert effort. Giving the agent \underline{u} can be viewed as the punishment phase of the equilibrium because this gives the agent the lowest possible payoff and the agent necessarily exerts low effort. If punishment is in the form of termination, then $f(\underline{u}) = \underline{v}$. We will show that when the minimum wage is binding but relatively low, the punishment is more efficiently done by suspending production for one period without termination the relationship. In this case $f(\underline{u}) > \underline{v}$.

Since the agent is paid wage \underline{w} in period 1 if $u_1 \in (\underline{u}, u_e]$, we introduce a function $L(u)$ which is defined by the following equation:

$$u = (1 - \delta)(\underline{w} - c) + \delta[(1 - p)L(u) + p(L(u) + k)]. \quad (3)$$

In other words, $L(u)$ corresponds to the agent's continuation payoff in the following period if he is paid \underline{w} this period, puts in effort, but the outcome is $Y = 0$.

For a relational contract that induces effort from the agent to be feasible, it is necessary that $L(u) \geq \underline{u}$. Therefore, the exact value of u_0 is given by $L(u_0) = \underline{u}$. Applying (??), we have

$$u_0 = (1 - \delta)(\underline{w} - c) + \delta(\underline{u} + pk). \quad (4)$$

This is summarized in Lemma 4.

Our next task is to derive the principal's highest equilibrium payoff $f(\underline{u})$ when the agent receives a payoff equals to his outside option. Note that there are two ways to lower the agent's utility to the level of his outside option. The first is simply to permanently terminate the relationship. In this case, the principal receives her outside option \underline{v} . Alternatively, when \underline{w} is sufficiently lower than \underline{u} , the principal may be able to give the agent an average payoff of \underline{u} by asking the agent to exert low effort for a period instead of terminating the relationship. By promising the agent a continuation payoff equal to $[\underline{u} - (1 - \delta)\underline{w}] / \delta$ regardless of the outcome, the agent's payoff will be kept at \underline{u} :

$$(1 - \delta)\underline{w} + \delta \frac{\underline{u} - (1 - \delta)\underline{w}}{\delta} = \underline{u}.$$

On such play path, the principal's average payoff will be

$$(1 - \delta)(qy - \underline{w}) + \delta f\left(\frac{\underline{u} - (1 - \delta)\underline{w}}{\delta}\right).$$

We call this punishment *temporary suspension of production* because the agent is expected to put in high effort after the punishment period. When $\underline{w} \geq \underline{u}$, clearly the agent's payoff can not be brought down to \underline{u} through temporary suspension of production. On the other hand, when \underline{w} is sufficiently low yet high enough that it binds, a convex combination of $(\underline{w}, qy - \underline{w})$ and $(\frac{\underline{u} - (1 - \delta)\underline{w}}{\delta}, f(\frac{\underline{u} - (1 - \delta)\underline{w}}{\delta}))$ which gives the agent \underline{u} can give the principal higher than \underline{v} . The following figure depicts how this arises:

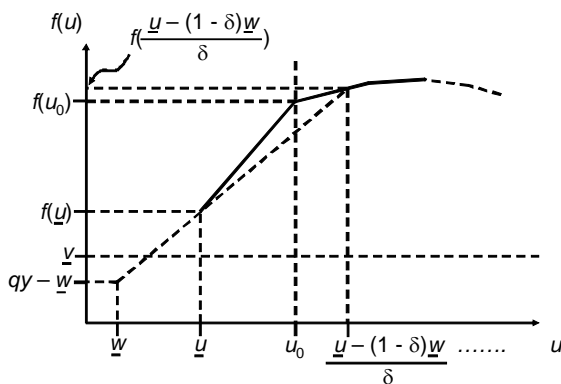


Figure 4: Determination of $f(\underline{u})$ when $f(\underline{u}) > \underline{v}$

In this case, it is more profitable for the principal to punish the agent by temporary suspension of production than by termination. In Lemma 5, we derive $f(\underline{u})$ and prove formally that there exists $w^* < \underline{u}$ such that termination dominates temporary suspension of production if and only if $\underline{w} > w^*$.

Next, we derive the exact value of u_e . Let u_r be defined by $L(u_r) = u_r$. From (3), we can see that $L'(u) = 1/\delta > 1$. Therefore, $L(u) < u$ if and only if $u < u_r$. It can also be verified that

$$u_r = L(u_r) = (\underline{w} - c) + \frac{\delta pk}{1 - \delta}.$$

When $L(u_r) + k \leq py - c - \underline{v}$, which can be rewritten as

$$py - \frac{[1 - \delta(1 - p)]c}{\delta(p - q)} \geq \underline{v} + \underline{w}, \quad (5)$$

it is clear that the following is an equilibrium: The game begins with the agent receiving a payoff of u_r . On the equilibrium path, the agent exerts effort every period and the principal offers a contract with $w_t = \underline{w}$ if $t = 1$ or if $Y_{t-1} = 0$ for $t > 1$, and offers a contract with $w_t = \underline{w} + k/(1 - \delta)$ if $Y_{t-1} = y$. Off the equilibrium path, the relationship is terminated forever. In this case, $u_e = u_r$. The efficient outcome can also be achieved for $u \in [u_e, u_{\max}]$ by raising the first-period wage. This is proved in Lemma 6a.

If $L(u_r) + k > py - c - \underline{v} \geq u_{\max}$, i.e., (5) is not satisfied, then the efficient outcome in which the agent exerts effort every period is no longer an equilibrium for the following reason. Note that $L(u) + k \leq u_{\max}$ must be satisfied for all u because the agent's continuation payoff cannot exceed u_{\max} . This implies that any equilibrium must start out with $u < u_r$. Since $L'(u) = 1/\delta$, following a bad outcome, it must hold that $L(u) < u$. More generally, if we let $L^T(u)$ denote the agent's continuation payoff following T bad outcomes, then $L^T(u) < L^{T-1}(u)$ and for T sufficiently large, $L^T(u)$ must fall below u_0 and termination or suspension of production must be triggered with a positive probability. Notice that it is more efficient to start out with a higher u because raising u both delays and lowers the probability of termination. Therefore, u_e is determined by $L(u_e) + k = u_{\max}$. The same level of efficiency achieved at $u = u_e$ can also be achieved for $u \in [u_e, u_{\max}]$ by raising the first-period wage. This is proved in Lemma 6b.

Recall that if $((u, f(u)))$ is on the Pareto frontier, then the continuation payoffs must lie on the Pareto frontier as well. In other words, the continuation payoffs corresponding to the good and bad outcomes must be $(L(u) + k, f(L(u) + k))$ and $(L(u), f(L(u)))$. This gives rise to the following functional equation that the Pareto frontier must satisfy:

$$f(u) = (1 - \delta)(py - \underline{w}) + \delta[pf(L(u) + k) + (1 - p)f(L(u))]. \quad (6)$$

This is stated as Lemma 7.

To summarize, the Pareto frontier of the PPE payoff satisfies

$$f(u) = \begin{cases} f(\underline{u}) + \frac{u - \underline{u}}{u_0 - \underline{u}}(f(u_0) - f(\underline{u})) & \text{if } u \in [\underline{u}, u_0] \\ (1 - \delta)(py - \underline{w}) + \delta[pf(L(u) + k) + (1 - p)f(L(u))] & \text{if } u \in [u_0, u_e] \\ f(u_e) + u_e - u & \text{if } u \in [u_e, u_{\max}], \end{cases} \quad (7)$$

where $u_0 = (1 - \delta)(\underline{w} - c) + \delta(\underline{u} + pk)$,

$$u_e = \begin{cases} u_r \equiv (\underline{w} - c) + \frac{\delta pk}{1 - \delta} & \text{if } u_r + k \leq py - c - \underline{v} \\ (1 - \delta)(\underline{w} - c) + \delta [u_{\max} - (1 - p)k] & \text{if } u_r + k > py - c - \underline{v}, \end{cases}$$

for some $u_{\max} \leq py - c - \underline{v}$ and

$$f(\underline{u}) = \begin{cases} (1 - \delta)(qy - \underline{w}) + \delta f\left(\frac{\underline{u} - (1 - \delta)\underline{w}}{\delta}\right) & \text{if } \underline{w} < w^* \\ \underline{v} & \text{if } \underline{w} \geq w^*, \end{cases}$$

for some $w^* < \underline{u}$.

The analysis is summarized in the following theorem.

Theorem 1 *There exist a unique $u_{\max} \leq py - c - \underline{v}$ (which holds in equality when $u_r + k \leq py - c - \underline{v}$) and a unique function f that satisfy (7) and f is the Pareto frontier of the PPE payoff set .*

Equation (7) expresses f as a solution to a functional equation. In general, existence and uniqueness of solutions to functional equations can be difficult to check. In our case, we can view the right hand side of equation (7) as an operator on f . It can be checked that this operator is monotone and nonexpansive, so it has a unique fixed point. In fact, we can even calculate f explicitly in some cases.

The conditions for the closed-form expression for f we consider here are

$$\begin{aligned} \underline{w} &\geq w^*, \\ u_r + k &\leq py - c - \underline{v}, \end{aligned}$$

and

$$\underline{u} + k = L(u_0) + k \geq u_e = (\underline{w} - c) + \frac{\delta pk}{(1 - \delta)}.$$

The first two conditions are familiar with the first implying that punishment is done through termination of the relationship and the second implying that the efficient outcome is sustainable as an equilibrium. The third condition is equivalent to

$$\underline{u} + \frac{(1 - \delta - \delta q)}{\delta(p - q)}c \geq \underline{w}, \quad (8)$$

which states that the expected payoff of the agent is greater than or equal to u_e following any good outcomes. And the condition is more likely to be satisfied when the probability of success (p) is small and when the discount factor (δ) is small. Corollary 1 gives the explicit formula and

Figure 5 illustrates.

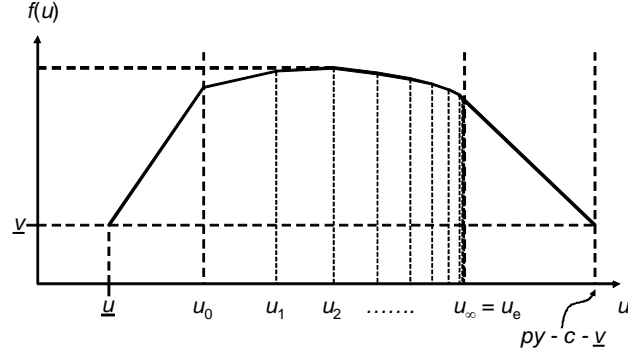


Figure 5: Pareto Frontier w/ Countable Number of Kinks

Corollary 1: If $\underline{w} \geq w^*$, $u_r + k \leq py - c - \underline{v}$, and (8) hold, then

$$f(u) = \begin{cases} \underline{v} + s_0(u - \underline{u}) & \text{if } u \in [\underline{u}, u_0] \\ \underline{v} + \frac{(u_0 - \underline{u})}{1 - \delta(1 - p)} \left(s_0 - \frac{p\delta(1 - \delta^{n+1})}{1 - \delta} - \delta^{n+1}s_n(1 - p) \right) + s_{n+1}(u - u_n) & \text{if } u \in [u_n, u_{n+1}] \\ f(u_e) + u_e - u & \text{if } u \in [u_e, u_e + f(u_e) - \underline{v}], \end{cases}$$

where

$$\begin{aligned} u_0 &= (1 - \delta)(\underline{w} - c) + \delta(\underline{u} + pk) \\ u_n &= u_0 + \frac{\delta(1 - \delta^n)}{1 - \delta}(u_0 - \underline{u}) \\ u_e &= (\underline{w} - c) + \frac{\delta pk}{(1 - \delta)} \\ s_0 &= \frac{(1 - \delta)(py - \underline{w}) + \delta((1 - p)\underline{v} + p(py - c - (\underline{u} + k))) - \underline{v}}{(1 - \delta)(\underline{w} - c) + \delta(\underline{u} + pk) - \underline{u}} \\ s_n &= s_0 - (1 + s_0)(p + (1 - (1 - p)^{n+1})) \\ f(u_e) &= py - c - ((\underline{w} - c) + \frac{\delta pk}{(1 - \delta)}). \end{aligned}$$

In Corollary 1, the middle region is partitioned by a sequence of thresholds $\{u_n\}_{n=1}^{\infty}$, and the Pareto frontier between any two adjacent points is a straight line. The thresholds satisfy that if the agent's expected payoff is u_n , his continuation payoff following a low outcome will move

to the left to u_{n-1} . This implies that for an agent with payoff u_n is guaranteed to stay in the relationship for at least n more periods: and he will be terminated if the next $n + 1$ outcomes are all low. As n goes to infinity, u_n converges to u_e , which is the lowest payoff at which the agent is given tenure.

Corollary 1 also illustrates a theoretical point of the non-differentiability of the Pareto frontier in repeated principal agent models. In particular, at each of the threshold u_n , the Pareto frontier f has a kink, and is thus not differentiable. The non-differentiability of the Pareto frontier has been pointed out by Thomas and Worrall (1994). Here we have an example with a countable number of nondifferentiable points. Interestingly, these nondifferentiable points converge to u_e , which happens to be a differentiable point with slope -1 , as the next subsection illustrates.

3.2 Optimal Relational Contract

The PPE payoff set provides a map for constructing relational contracts. In particular, every point in the PPE payoff set specifies the actions to be taken in the current period and the continuation payoffs (associated with the outcomes) that again lies inside the PPE payoff set. Any history of play leads to a continuation payoff for the players, who then choose their actions associated with the continuation payoffs. Theorem 1 specifies the actions taken on the Pareto frontier. Moreover, it implies that any payoff on the Pareto frontier can lead to continuation payoffs that also belong to the Pareto frontier. In other words, take any point $(u, f(u))$ on the Pareto frontier, we can use Theorem 1 to construct a relational contract that gives the agent u and the principal $f(u)$. To solve for the optimal relational contract, we need to identify the agent's payoff u that maximizes $f(u)$.

The key result in this subsection states that the optimal relational contract is always inefficient. In other words, for any discount factor, there is positive probability that either the relationship terminates or the worker shirks (but not both). It is easy to see why this is the case for the low surplus case. Proposition 1 states the result formally for the high surplus case.

Proposition 1: *Suppose (5) holds. Then $f'(u_e) = -1$ and there exists $u < u_e$ such that*

$$f(u) > f(u_e).$$

Proof. Because f is concave, then for almost all $u \in [u_0, u_e]$, Theorem 1 implies that

$$f'(u) = pf'(L(u) + k) + (1 - p)f'(L(u)). \quad (9)$$

Take $u'_0 < u_e$ such that $f(u'_0)$ is differentiable and $L(u'_0) + k > u_e$. Take u'_1 such that $L(u'_1) = u'_0$. According to (9) and the fact that $u'_0 < u_e < u'_0 + k$, within the interval of $[u'_0, u_e)$,

we can find u'_1 such that

$$f'(u'_1) \leq -p + (1-p)f'(u'_0).$$

This procedure can continue forever, i.e., we can find $u'_{n+1} \in (u'_n, u_e)$ such that

$$f'(u'_{n+1}) \leq -p + (1-p)f'(u'_n).$$

Moreover, for any u'_n such that $f'(u'_n) = -1 + \varepsilon$, there exists $u'_{n+1} \in (u'_n, u_e)$ such that

$$-1 + \varepsilon > f'(u'_{n+1}) > -1.$$

Therefore, $f'(u_{e-}) = \lim_{n \rightarrow \infty} f'(u'_n) = -1$.

Once we establish that $f'(u_e) = -1$, then it is immediate that there exists $u < u_e$ such that $f(u) > f(u_e)$. ■

From Theorem 1, we know that the right derivative of $f(u_e)$ is -1 . Proposition 1 essentially says that when (5) holds, the left derivative of $f(u_e)$ is also -1 . In other words, at $u = u_e$, the principal has a strict incentive to lower the agent's payoff in the sense that for every dollar of payoff taken away the agent, the principal gains one dollar. This implies optimality requires that the principal to set $u < u_e$. Since u_e is the smallest payoff of the agent that the joint surplus is maximized, Proposition 1 then implies that the optimal relational contract does not maximize the joint surplus when (5) holds. When the limited liability is mildly binding ($\underline{w} < w^*$) the inefficiency takes the form of worker shirking in some periods. When the limited liability is sufficiently binding ($\underline{w} > w^*$), termination occurs.

The reason for inefficiency in Proposition 1 reflects the tension between efficiency and rent extraction. In any efficient relational contract, the relationship never terminates, so the agent is entirely motivated through bonus and obtains rent every period. One way to reduce the agent's rent is to substitute bonus with future rent as reward: high output increases the chance that the worker will be asked to put in effort in the future (and thus receives rents) and low output reduces it. By doing so, the principal saves the bonus payment to the worker, but it contains a cost. In particular, since luck matters, outputs can be low even if the agent always put in effort. And low past outputs can result in the worker not being asked for effort in some periods, resulting in inefficiency. In other words, the principal may benefit from the possibility of having inefficiency to extract more rent from the worker.

On the other hand, when inefficient actions are taken, it is also costly to the principal. So, *a priori*, it is not clear it is profitable for the principal to risk terminating the relationship to save on bonus payments. Indeed if we restrict the contracts to be stationary, the cost from inefficiency can be sufficiently high that the optimal relational contract within this class can be efficient. The actual optimal relational contract is nonstationary and is inefficient. In particular, by moving the

agent’s payoff just a bit to the left of u_e , inefficiency can arise only in some arbitrarily discounted future with very small probability. In other words, the marginal cost of using inefficiency at u_e is zero for the principal, and there is positive marginal benefit through lowered bonus payments. In the optimal relational contract, the principal move the agent’s payoff further to the left until the marginal benefit is equal to the marginal cost, i.e., every dollar of payoff he takes away from the agent via lowered bonus is completely cancelled by the increased loss of surplus in the relationship via inefficient actions. This implies that the agent’s payoff in the optimal relational contract will fall into the middle region.

4 Empirical Implications

This section studies the empirical implications of the optimal relational contract. As explained in Section 3, termination arises in equilibrium if the limited liability constraint is sufficiently binding. We explore in detail the termination dynamics and the associated compensation dynamics in this section. To this end, we assume below $\underline{w} \geq \underline{u}$, so the limited liability is sufficiently binding. In Subsection 4.1, we describe the employment dynamics when the surplus in the relationship is high. In Subsection 4.2, we examine the low surplus case. In Subsection 4.3, we discuss how the firm’s ability to commit to long-term contracts affects employment dynamics.

4.1 Employment Dynamics with High Surplus

In this subsection, we examine the employment dynamics resulting from the optimal relational contract when the relationship has sufficiently high surplus. Specifically, we assume Equation (5) holds. In this case, the worker starts the employment relationship in a “probation phase” in which he receives \underline{w} regardless of his performance, and depending on his performance, he either receives “tenure” or is terminated.¹⁵ Proposition 2 characterizes the employment structure formally.

Proposition 2: *In the optimal relational contract, the set of histories can be partitioned into $H = H_1 \cup H_2 \cup H_3$, such that*

(i): *(probation phase): When $h^t \in H_1$, $w(h^t) = \underline{w}$.*

(ii): *(termination phase): When $h^t \in H_2$, both the principal and the agent receive their outside options $(\underline{u}, \underline{v})$.*

(iii): *(tenure phase): When $h^t \in H_3$, the optimal relational contract can be implemented in*

¹⁵We use the terms of probation and tenure for expositional ease. There is no explicit formal contract in this model specifying that the worker is in the probation period or is guaranteed lifetime employment.

the following way:

$$\begin{aligned} w(h^{t+1}) &= \underline{w} && \text{if } y_t = 0; \\ &= \underline{w} + k && \text{if } y_t = y. \end{aligned}$$

In other words, optimal relational contract in H_3 may be stationary.

(iv): Employment starts with the probation phase:

$$h^1 \in H_1.$$

(v): Phase 2 or 3 are both absorbing. More specifically, if $h^t \in H_i$, for $i = 2, 3$, then $h^{t+k} \in H_i$ if $h^{t+k}|t = h^t$.

$$\lim_{t \rightarrow \infty} \Pr(h^t \in H_2 \cup H_3) = 1.$$

Proof. Define H_3 as the set of histories such that the agent's continuation payoff $u \geq u_e$. Define H_2 as the set of histories such that the agent's continuation payoff $u = \underline{u}$. By Theorem 1, it is clear $H_2 \cap H_3 = \Phi$. Define $H_1 = H \setminus (H_2 \cup H_3)$.

By Proposition 1, we know that the game starts in H_1 , so (iv) is proved. Theorem 1 also directly gives (i). Since $(\underline{u}, \underline{v})$ is the unique PPE payoff in which the agent's payoff is \underline{u} and it is supported by taking outside option forever, (ii) is proved. Now if $u \geq u_e$, then $f(u) + u = py - c$, so the continuation payoff must be $py - c$ as well. Moreover, $(u, f(u))$ can be implemented by paying the agent $\underline{w} + \frac{u - u_e}{1 - \delta}$ in this period, and uses $(u_e, f(u_e))$ and $(u_e + k, f(u_e) - k)$ as continuation payoffs forever. This proves (iii). Finally, take any u , there exists an N such that with a fixed probability bounded away from 0, the agent either have continuation payoff (u, v) or $u \geq u_e$. Then (v) follows from standard statistics arguments. ■

The three phases in the employment dynamics correspond to the three regions in the PPE payoff set. Proposition 1 implies that the optimal relational contract starts in the middle region, and the continuation payoffs vary according to the outcomes. If the outcome is good, the continuation payoff moves to the right. Otherwise, it moves to the left. When the continuation payoff moves across the right threshold (u_e) the agent receives permanent employment, and the continuation optimal contract can be implemented in a sequence of stationary contracts (cf Levin (2003)). When the continuation payoff crosses the left threshold (u_0), then termination occurs with positive probability.

The employment structure from Proposition 2 has several key features. Most notably, termination arises endogenously. This stands in contrast with early efficiency wage models in which there is no production uncertainty in equilibrium and termination does not occur. One reason for the difference is the stochastic production in our model. Failures in performance can arise in

equilibrium even if the worker always put in effort. When failures occur sufficiently often, the firm has to terminate the worker (even if it knows in equilibrium the worker has always put in effort) because not doing so will give the worker incentive to shirk. Such equilibrium punishment is a common feature of repeated game with imperfect public monitoring; see Green and Porter (1984) for a seminal paper in the literature.

While termination will be used as a tool to induce incentive, Proposition 2 implies that the firm will at some point give up this tool by granting tenure. This feature of tenure underlies the key intuition of how rewards are structured to solve a dynamic moral hazard problem with limited liability. In particular, when choosing between rewarding the agent with monetary award or future job security, the firm should *frontload* the reward with job security. Different from giving out bonus, rewarding with future job security has the benefit of better preserving the future surplus of the relationship by increasing the probability that the relationship will last. In other words, monetary award only affects how the pie is divided, yet giving out job security increases the size of the pie. This reasoning implies that the worker is rewarded with job security in the earlier stage of the employment relationship, and performance bonuses are paid out only after the worker has received tenure (when the surplus in the relationship is sufficiently large). This reward structure sheds light on a number of features of the labor market. We discuss its implications on turnover and compensation dynamics separately.

Turnover Dynamics Our model applies to involuntary turnover due to poor performance. Proposition 2 implies that the employment relationship starts with a probation phase, during which involuntary turnover can occur. Probation is common across jobs and is a salient feature in professions characterized by up-or-out contracts. Typical explanations for why probation exists include learning about workers, screening out bad workers, and encouraging workers to invest in firm-specific human capital. Our model suggests that probation period can also arise as part of the optimal relational contract when the job has rents. The firm extracts as much rent as possible by having a probation period during which the workers are paid lower wages.

This rent-extraction explanation appears to fit well with the use of probation period in Henry Ford’s five-dollar-day Program, in which the workers are paid more than twice of the prevailing wage. To qualify for this program, a worker is required to have worked for Ford for at least a six-month probation period during which they receive the prevailing market wage. Raff and Summers (1989) argue the use of probation in this example is consistent with the efficiency wage hypothesis which sometimes involve deferred compensations.¹⁶ Explanations for probation based on learning, screening, or improving worker qualities are less relevant in this context because

¹⁶On the other hand, we are unaware of efficiency wage models that explicitly generate the probation period. In fact, the reason that wage is deferred in this model is different from earlier efficiency wage models. We discuss the difference in the subsection on compensation dynamics.

the majorities of the jobs at Ford during that time are simple and routine. Raff and Summers (1989) provide the following cite from Meyer (1981), pp 41:

"Division of labor has been carried to such a point that an overwhelming majority of jobs consist of a very few simple operations. In most cases a complete mastery of movements does not take more than from five to ten minutes. All the training that a man receives in connection with his job consists of one or two demonstration by his foreman or a worker who has been doing the job. After these demonstrations he is considered a fully qualified "production man." All that he has to do now is to autonomize these few operations now so that speed may rapidly be increased."

While the rent extraction rationale sheds light on the use of probation, the predicted "probation phase" is in general stochastic in length. Corollary 2 shows that sometimes, however, the predicted probation phase has a fixed duration.¹⁷

Corollary 2: *When $\underline{u} + \frac{(1-\delta-\delta q)}{\delta(p-q)}c \geq \underline{w}$, there exists T^* such that the turnover rate is 0 for $t < T^*$ and is again 0 for $t > T^* + 1$. Generically, turnover happens only in T^* .*

Proof. When $\underline{u} + k \geq (\underline{w} - c) + \frac{\delta pk}{(1-\delta)}$, Corollary 2 gives an explicit formula of f . There are two cases to consider. In case 1, there's a unique u_n that maximizes $f(u)$. In this case, if any of the output in the first $n + 2$ periods is positive, the agent receives permanent employment. Otherwise, the agent's continuation payoff moves to u_{n+1-t} in period t , and is terminated at time $t = n + 2$.

In case 2, there exists n such that $f(u)$ is maximized in $[u_{n-1}, u_n]$. In this case, if no positive outcome has been generated, the agent's continuation will be in $[u_{n-t}, u_{n+1-t}]$ in time t . And the agent will be terminated either in time $t = n + 1$ or $t = n + 2$. ■

The condition in Corollary 2 is satisfied when the probability of success is small. In this case, there is a fixed period at which the worker is terminated, and if not, the worker will be granted tenure. Such turnover process resembles the "up-or-out" rule, which is frequently used in professional jobs. In particular, Corollary 2 appears to fit some segment of the academic labor market where successful performance can be difficult: a few home-run publications bring an assistant professor to tenure, and failure to do so before a fixed date leads to termination.

The turnover process in our model has an interesting comparison to the up-or-out rule in O'Flaherty and Siow (1992, 1995). O'Flaherty and Siow develop a model of symmetric learning with assignment: the worker is initially assigned to a bottom job, and his performance on the job reveals his ability over time. Assignment to the bottom job corresponds to the probation

¹⁷Another case in which the probation phase has fixed duration is when $p = 1$. But there is no involuntary turnover in this case.

phase. The worker is promoted when his expected ability exceeds a threshold and is terminated when it falls below another (lower) threshold. Just as in our model, failure on the job increases the chance of termination and success decreases it, and the duration of probation is in general stochastic. More interestingly, O’Flaherty and Siow also show that the duration of probation can be fixed. In such case, the worker is out if one failure occurs and is promoted only if no failure appears during the entire probation phase. In our model, in contrast, the worker is "promoted" if one success occurs and is terminated only if no success appears during the probation.

Just as in O’Flaherty and Siow (1992, 1995), models on the dynamics of performance-related involuntary turnover typically rely on learning about the workers; see for example Jovanovic (1979). Our analysis suggests that moral hazard models can complement these models in understanding this issue. Some predictions of this model is similar to those of learning models. For example, we predict that older workers are less likely to be fired for poor performance.¹⁸ More interesting, the turnover rate can also be inverse-U shaped with respect to employment duration:¹⁹ Corollary 2 gives a degenerate inverse-U; Turnover rate is positive in period T^* but is zero both before and after period T^* . The reason that turnover rates may rise initially reflects how the optimal relational contract minimizes the loss of surplus from termination. In particular, the principal would like to postpone using termination as much as he can while maintaining incentive from the agent. This postponement of termination implies that the turnover rate can be low at the beginning of the employment relationship.

Other predictions of the model, however, are different from learning models. For example, tenure and termination typically does not arise simultaneously in learning models. In addition, this model predicts a type of path-dependency absent in learning models.²⁰ More importantly, learning models will lead to different compensation dynamics, which we discuss next.

Compensation Dynamics The optimal relational contract leads to two predictions of the compensation dynamics. First, compensation is deferred in the sense that average pay is higher once the worker receives tenure. This follows directly from Proposition 2: the average pay in the probation phase is \underline{w} and is $\underline{w} + pk$ in the tenure phase. The logic for this is straightforward: during the probation phase, the firm substitutes bonus with job security, which results in lower average compensations. Note that for a single worker, the model predicts deferred compensation between phases and not within. In fact, the model does not even pin down a unique earning path

¹⁸Chevalier and Ellison (1999) find that termination is more performance sensitive to younger mutual fund managers.

¹⁹This is the celebrated prediction of Jovanovic (1979) and appears to fit the data; see for example Farber (1994)

²⁰In a stable learning environment, the update of the worker’s ability is unaffected by the sequence of outputs. In other words, a high output followed by a low output leads to the same estimate (of ability) as a low output followed by a high output. In our model, in contrast, the worker’s continuation payoff will be higher in the former case. This is because the reward of the agent is deferred.

within the tenure phase.²¹ At the aggregate level, since the duration of the probation phase can be stochastic, the average earning rises with time on the job. Moreover, the increase in earning eventually goes to zero as most workers (who remain in the firm) are tenured.

The upward sloping earning profile is a well-documented fact and can be explained by many theories; see Rubinstein and Weiss (2007) for a recent survey. In several important earlier papers, it is shown that the rising earning profile can be explained by dynamic moral hazard models: see for example Lazear (1979, 1981), and Akerlof and Katz (1989). The mechanism is different here. In Lazear (1979, 1981), compensation is deferred to solve the moral hazard problem at the end of a worker's career. There is no limited liability and workers may not have rents in his models. When limited liability is present, Akerlof and Katz (1989) stress the importance of having rent in the relationship so as to induce effort from the worker at the beginning of his career. In their model, the production is deterministic (on the equilibrium path), and the firm can commit to long-term contracts, so there is great flexibility on how wages are paid out. Different from Akerlof and Katz (1989), we take the existence of rent as given and focus on the implications of optimal rent extraction on earning and turnover dynamics when the production function is stochastic.

The model's more novel implication centers on the sensitivity of compensation to performance. The model implies that compensation is insensitive to performance in the worker's earlier career and it becomes more sensitive to performance only after tenure. This happens because reward for good performance in the worker's earlier career takes the form of increased job security instead of bonus payment. We think that this is a plausible mechanism, and existing (indirect) evidence appears to be supportive in general. For example, Hashimoto (1979) finds that the bonus to wage ratio is increasing with experience in Japanese firms. Gibbons and Murphy (1992) show that the pay of older CEOs are more sensitive to stock market performance. Gompers and Lerner (1999) document that the sensitivity of compensation to performance in the second fund is higher than the first one. Misra, Coughlan and Narasimhan (2005) find that salary to total compensation ratio is decreasing with the seniority of salespeople. On the other hand, Khan and Sherer (1990) find bonus are more sensitive to performance for managers with lower seniority.²²

Our predictions on compensation dynamics differentiate this model from the learning models. Absent of other factors, learning models predict that the expected earning does not change over time within a job (Jovanovic (1979), Farber and Gibbons (1996)). Moreover, the earning

²¹The firm can reallocate the bonus across periods as long as he will not renege on them. This indeterminacy of earning path maybe resolved if we assume that the worker is risk averse or has a different discount factor from the firm.

²²Different from the rest of the papers cited above, the finding of Kahn and Sherer (1990) is based on a single firm. Workers in this firm has considerable job security: the annual discharge rate is 0.5%, one tenth of the industry average.

becomes less sensitive to performance over time as more information is known about the worker. When other factors are included, learning models can explain one of the two earning patterns, but typically not both.²³In addition, these models do not give joint implications on turnover and employment dynamically. Finally, the learning models in general do not explore how the employment dynamics is affected by the surplus in the relationship, as we come to next.

4.2 Employment Dynamics with Low Surplus

In this subsection, we characterize the turnover and compensation dynamics when the surplus is low. Specifically, we assume (5) fails so it is impossible to sustain an equilibrium that achieves the first best. In this case, the worker will never be given tenure, and the relationship terminates with probability 1. This result, in comparison with Proposition 2, sheds light on why and where lifetime employment occurs and its implications on compensation dynamics.

The following proposition formally state our result on turnover and compensation dynamics. In contrast to the high surplus case (where compensation dynamics becomes indeterminate once tenure is granted), there is no ambiguity in the entire compensation dynamics. In equation (10), $\underline{w} + pk$ is the worker's average compensation upon tenure. Proposition 3 states that conditional on remaining in the relationship, the worker's compensation in the long run is lower in the low-surplus case.

Proposition 3: *If $py - \frac{[1-\delta(1-p)]c}{\delta(p-q)} < \underline{v} + \underline{w}$, then the followings are true:*

(i): *The relationship terminates with probability 1:*

$$\lim_{t \rightarrow \infty} \Pr(u_t = \underline{u}) = 1.$$

(ii): *Period 1 wage is given by*

$$w_1 = \underline{w}.$$

The wage at the beginning of period t for an agent with expected payoff $u_t > u_0$ is given by

$$w_t = \underline{w} + \max\{u_t - u_e, 0\},$$

where u_e is determined by Theorem 1, and

$$E[w_t | u_t > \underline{u}] < \underline{w} + pk. \tag{10}$$

²³For example, Harris and Holmstrom (1982) is a learning model that generates upward sloping earning profile when assuming the worker is risk averse and the firms can commit to long-term contracts. But earning becomes less sensitive to performance over time. Gibbons and Murphy (1992) show that the firm should optimally design contracts to make pay more sensitive to performance over time in order to overcome the weakening incentive for effort from career concern. But it's not clear how average earning change over time in their model.

In this low-surplus case, tenure cannot be part of the equilibrium. This is because if the worker receives tenure, bonus becomes the firm's only instrument to induce effort. But the bonus required to induce exceeds the firm's future payoff from the relationship (in this low-surplus case) and the firm will renege. Consequently, to induce effort from the worker, apart from rewarding success with the maximum bonus the principal is willing to pay, it is necessary to punish failure with (increased future probability of) termination. The absence of tenure implies that a sufficiently long sequence of failures will lead to termination. Since such long-streaks will occur almost surely as time goes to infinity, the relationship is terminated with probability 1.

While the patterns of dynamics differ between the low and high surplus case, the underlying logic for the employment structure remains the same. In particular, when choosing between bonus and job security as reward, the firm will again frontload job security as much as it can. In the low surplus case, however, the frontloading is incomplete (in the sense that no tenure can be given) since the firm needs to use the threat of job insecurity to help induce effort.

The incomplete frontloading of job security has two implications on compensation dynamics. First, the compensation remains deferred to some degree. In particular, while there is not a distinct probation phase, the compensations are low and are insensitive to performance in the earlier stage of employment. Second, the expected long-run compensation is lower in the low-surplus case (see equation (10)). This is because in the long run, almost all of the workers in the high-surplus relationship have tenure, and the reward for good performances are all given by bonus. In the low-surplus relationship, none of the workers have tenure, so a fraction of the reward is given by future job security. This results in a smaller bonus on average than the high-surplus case and lower over all compensation.

In terms of turnover dynamics, Proposition 3 suggests that lifetime employment is more likely to arise in high-surplus firms. Casual empiricism appears to support this prediction. Both Hewlett-Packard and IBM are prominent examples of firms promising employment security when business conditions are favorable and have to abandon it when the business conditions become challenging (Rogers and Beer (1995) and Mills and Friesen (1996)). There is also some indirect scientific evidence. For example, if we take the view that Japanese employment relationship has higher surplus than the U.S., possibly due to higher levels of firm-specific human capital, then our prediction is consistent with the finding that lifetime employment is more prevalent in Japan (Hashimoto and Raisian (1985)).²⁴

Proposition 3 also sheds light on how compensation dynamics varies with the surplus in the relationship. Comparing the compensations in the long run, it points to a few testable differences between the high and low surplus firms. In particular, as surplus in the relationship increases,

²⁴Waldman (2007) reviews the facts and models on U.S-Japan difference.

our results suggest that a) the expected compensation is higher; b) the earning profile is steeper; c) the bonus to wage ratio is higher. There appears to be some related evidence on the first two points. In an influential paper, Abowd, Kramarz, and Margolis (1999) finds that controlling for personal effect, firms that pay higher wages are also more productive and profitable. They also report larger heterogeneity of returns to seniority across firms, but we do not know how the slopes vary with profitability. At a cross-country level, if we again take the view that Japanese employment relationship has higher surplus, the evidence appears to be supportive: Hashimoto and Raisian (1985) estimate that earning growth due to experience on the job is significantly higher in Japan. Finally, we are not aware of papers that relates the bonus to wage ratio with firm profitability, and it will be interesting to examine the relationship of the two and also the joint relationship among firm profitability, earning level, earning slope, bonus to wage ratio, and job security.

4.3 Commitment

In this subsection, we examine how the employment dynamics change when the firm can commit to long-term contracts. Firm's ability to commit does not affect the employment dynamics in the high-surplus case. In the low-surplus case, permanent employment occurs with positive probability when the firm can commit. Upon tenure, the optimal long-term contract with commitment can be implemented as a sequence of stationary contracts just as in the high-surplus case. To the extent that larger firms have a larger commitment power, by virtue of forming relational contracts with more workers, our results suggest that the expected compensation, the return to seniority, and the bonus to wage ratio are all higher in larger firms. Theorem 2 characterizes the Pareto frontier of the long-term contracts under full commitment when the limited liability is sufficiently binding.

Theorem 2: *Suppose $\underline{w} \geq \underline{u}$. When the firm can commit to long-term contracts, the Pareto frontier is given by the unique function that solves the following equation*

$$f(u) = \begin{cases} \underline{v} + \frac{u-\underline{u}}{u_0-\underline{u}}(f(u_0) - \underline{v}) & \text{if } u \in [\underline{u}, u_0] \\ (1-\delta)(py - \underline{w}) + \delta[pf(L(u) + k) + (1-p)f(L(u))] & \text{if } u \in [u_0, u_e] \\ f(u_e) + u_e - u & \text{if } u \in [u_e, u_e + f(u_e) - \underline{v}], \end{cases} \quad (11)$$

where $u_e = (\underline{w} - c) + \frac{\delta pk}{(1-\delta)}$, $f(u_e) = py - c - ((\underline{w} - c) + \frac{\delta pk}{(1-\delta)})$, and $u_0 = (1-\delta)(\underline{w} - c) + \delta(\underline{u} + pk)$.

In particular, the set of histories can be partitioned into $H = H_1 \cup H_2 \cup H_3$, such that

(i): (H_1 is the probation phase): When $h^t \in H_1$, $w(h^t) = \underline{w}$.

(ii): (H_2 is the termination phase): When $h^t \in H_2$, both the principal and the agent receive their outside options $(\underline{u}, \underline{v})$.

(iii): (H_3 is tenure phase. Optimal long-term contract in H_3 may be stationary): When

$h^t \in H_3$, the optimal long-term contract can be implemented in the following way:

$$\begin{aligned} w(h^{t+1}) &= \underline{w} && \text{if } y_t = 0; \\ &= \underline{w} + k && \text{if } y_t = y. \end{aligned}$$

(iv): (Employment starts with the probation phase)

$$h^1 \in H_1.$$

(v): (Phase 2 or 3 are both absorbing): If $h^t \in H_i$, for $i = 2, 3$, then $h^{t+k} \in H_i$ if $h^{t+k}|t = h^t$.

$$\lim_{t \rightarrow \infty} \Pr(h^t \in H_2 \cup H_3) = 1.$$

Proof. When the agent's expected payoff is u_e , the long-term contract cannot do better than $py - c - u_e$. The rest of the theorem is proved in exactly the same way as Theorem 1. The rest follows just as in Proposition 3. ■

Note that the equations on the Pareto frontier are identical to the high-surplus case in Theorem 1. This implies that the commitment does not affect the employment dynamics when the surplus is high: the worker starts in the employment relationship in a probation phase, and depending on the outputs, he will either receive permanent employment or be terminated. The only caveat is that upon receiving tenure, there is more flexibility on how the bonus are paid out: when the firm can commit, the bonus payment needs not be bounded by the surplus in the relationship.

In the low-surplus case, Theorem 2 implies that the employment dynamics associated with the optimal-long-term contract is similar to the one in the high-surplus case and is thus different from the case without commitment. This difference can be explained by the firm's ability to frontload job security. With long-term contract, the firm can commit to give bonus to the worker even if the size of the bonus is larger than the firm's expected future surplus. This gives the firm extra freedom to push back using bonus as reward (or better frontload using job security as reward). Just as in the high-surplus case, the worker will not receive bonus payment until the tenure phase.

In summary, commitment does not affect employment dynamics in the high-surplus case. When the surplus in the relationship is low, changes in employment dynamics due to commitment is similar to that due to surplus increases. One proxy for the firm's ability to commit is its employee size: suppose all employees will punish the firm if it reneges on one worker, then as

the number of employee increases, the firm has more to lose and is thus less likely to renege.²⁵ In this case, the model suggests that as the firm size increases, a) lifetime employment is more likely to occur; b) compensation is higher; c) earning profile is steeper, and d) wage to bonus is higher in the long run. Moreover, these effects will be more pronounced for firms with low profitability.

Our prediction sheds light on the empirical literature on firm-size effects. In terms of turnover dynamics, Idson (1996) reports that the layoff rate decreases with firms size and average tenure increases with it. For firms with fewer than 24 employees, the annual layoff rate is 9.2% and the tenure is 7.0 year. The corresponding figures become 2.6% and 11.0 years in firms with 1000+ employees. It is also well-known that larger employers pay higher wages; see for example Oi and Idson (1999).²⁶ In addition, studies in general find steeper slopes of earning profile at larger firms; see for example Hashimoto and Raisian (1985), Pearce (1990), and Reilly (1995). Finally, Hashimoto (1979) reports higher bonus to wage ratio in larger Japanese firms. Our model suggests that these patterns are related to larger firm's ability to better defer compensation. Moreover, these patterns should be stronger when restricted to firms with lower profitability. Our results suggest further empirical tests in this area that focus on the interaction term of firm size and its profitability.

5 Conclusion

This paper develops a tractable model of relational contracts of imperfect public monitoring with limited liability. The optimal relational contract highlights key features of the principal's extraction of rent from the agent and the optimal use of job security and bonuses to provide incentive to exert effort. Optimal relational contracts in standard repeated principal-agent relationships tend to generate very few concrete predictions because the principal and the agent can freely make transfers. In contract, the combination of imperfect monitoring and limited liability in our model generates a number of patterns of turnover and compensation dynamics that are largely consistent with empirical findings.

First, employment relationships tend to begin in a probation period during which the agent is paid a flat wage. Second, wage is more sensitive to performance over time and incentive pay is deferred until the agent has sufficient job security or until the principal's incentive to renege is impending. Third, our model predicts that high surplus firms provide more job security, higher

²⁵To prevent the firm from renegeing from all of the workers simultaneously, one caveat is that the outputs of the workers cannot be perfectly positively correlated. See Levin (2002) for an analysis of the optimal multilateral relational contracts.

²⁶Brown and Medoff (1989) considers several explanations for the employer size-wage effect. They find that higher worker quality is an important factor but much of the wage premium remain unexplained. Abowd, Kramarz, Margolis (1999) estimate that about 75% of this effect can be attributed to worker fixed effects.

average pay, steeper earning profile, and higher sensitivity to pay to performance. Finally, our results imply that a firm's ability to commit has similar effect of high surplus from the relationship, but this is true only when a relationship has low surplus. In other words, ability to commit is substitute of high surplus when the surplus in the relationship is insufficient.

The comparison of our main findings in the optimal relationship contracts with those in the optimal long-term contract also allows us to bridge the literature of relational contracts with the literature of dynamic contracting. The finding that firm's ability to commit does not impact the optimal relational contract when the surplus in the relationship is high suggests that these two literatures are indeed closely related. On the other hand, when the surplus from the relationship is low, there is qualitative difference in the termination dynamics, which is especially prominent when the minimum wage is not too low. In this case, periodic punishment in the form of temporary suspension of production perpetually recur in the optimal relational contract but production become eventually efficient in the optimal long-term contract.

The tractability of the model also implies that some interesting margins of adjustment are not explored in this model. For example, if the agent can put in multiple levels (or continuum) of effort, then one may study how the agent's effort choice is affected by the history of outputs. With multiple effort levels, the basic lesson that job security should be frontloaded and bonus should be postponed as much as possible still remains valid. But it appears difficult to state something general about how effort evolves over time.²⁷

Finally, it may be interesting to embed this model into a general equilibrium framework, so that the outside options of the workers and firms are endogenized. This model has potential to generate multiple equilibrium in turnover patterns. In particular, the surplus in the relationship depends on the easiness to form new employment relationship. When new employment relationship is hard to form, the surplus in the existing relationship is high. Our results then suggest that the worker is more likely to receive tenure, making it even harder to form new employment relationship in the labor market. Such multiplicity of equilibrium may help shed light on large cross country differences in employment patterns.

²⁷In models where the principal can commit, effort level may be nonmonotone in the agent's continuation value; see for example Spear and Srivastava (1987) and Clementi and Hopenhayn (2006).

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6 Appendix

First, we show that the set of PPE payoff is completely characterized by the Pareto frontier.

Lemma 0: Let u_{\max} be the maximum PPE payoff of the agent. Then the PPE payoff set E is given by

$$E = \{(u, v) : u \in [\underline{u}, u_{\max}], v \in [\underline{v}, f(u)]\}$$

Proof. First, note that the payoff pair $(\underline{u}, \underline{v})$ (meaning that the agent's normalized expected payoff is \underline{u} and the principal's normalized payoff is \underline{v}) is in the PPE payoff set. This payoff is supported by an equilibrium in which on the equilibrium path, the principal and the agent does not start a relationship, and off the equilibrium path, the agent never puts in effort and both the principal and the agent do the start the relationship in the future.

Second, it follows by convexity of the PPE payoff set that any payoff on the line segment between $(\underline{u}, \underline{v})$ and $(\underline{u}, f(\underline{u}))$ can be supported as a PPE payoff, and this is the left boundary of the PPE payoff set. Third, because there is no limit in the amount of transfer the principal can make to the agent at the beginning of period 1, it is easily seen that the lower boundary of the PPE set is given by the horizontal line at \underline{v} . Finally, convexity implies that any equilibrium payoff between (u, \underline{v}) and $(u, f(u))$ can be obtained. ■

Now we follow seven steps to prove Theorem 1 which characterizes the Pareto Frontier. First, we show that there exists a threshold u_e such that f has a slope of -1 to the right of u_e (Lemma 1) and has a slope larger than -1 to the left of u_e (Corollary 1a). Second, we show that to the left of u_e , only the minimum wage \underline{w} will be paid out regardless of the output (Lemma 2). Third, we show that there exists a threshold u_0 (the value of which is determined in Lemma 4) such that to the left of u_0 , f is a straight line between $(\underline{u}, f(\underline{u}))$ and $(u_0, f(u_0))$ (Lemma 3). Following a sequence of bad outcomes, both the principal and agent will be punished. Our fourth step is to show that there exists a threshold w^* such that if $\underline{w} \geq w^*$, then punishment will be in the form of termination in which case the principal receives $f(\underline{u}) = \underline{v}$ and if $\underline{w} < w^*$, then punishment will be done through temporary suspension of production and in which case the principal receive $f(\underline{u}) > \underline{v}$ (Lemma 5). Fifth, we determine the value of u_e (Lemmas 6a-6b). Sixth, we show that the value of f between u_0 and u_e is the fixed point of a contraction mapping indexed by the value of $f(u_0)$ (Lemma 7). Steps 1-6 presumes the existence of f . In Step 7, we establish the existence and uniqueness of $f(u)$.

Denote u_e as the smallest PPE payoff of the agent that maximizes the sum of the equilibrium payoffs of the principal and agent. More precisely, define $M := \{u : f(u) + u \geq f(u') + u'\}$ and $u_e := \{u \in M : u \leq u' \forall u' \in M\}$. First, $(u_e, f(u_e))$ exists because the payoff set is compact and f is continuous in the compact set. Next, we show that

Lemma 1 $f(u_{\max}) = \underline{v}$ and $f(u) = f(u_e) + u_e - u$ for $u \in [u_e, u_{\max}]$.

Proof. First, we establish that $f(u_{\max}) = \underline{v}$. Suppose $f(u_{\max}) > \underline{v}$. Then the principal can increase the first period transfer to the agent by $f(u_{\max}) - \underline{v}$ and raise the agent's payoff to $u_{\max} + f(u_{\max}) - \underline{v}$. This contradicts the definition of u_{\max} .

Notice that $u + f(u) \geq u_e + f(u_e)$ for $u > u_e$ because principal can freely pay the agent extra in the first period; and $u + f(u) \leq u_e + f(u_e)$ holds by definition of u_e . This completes the proof. ■

The following corollary follows Lemma 1, definition of u_e , and concavity of f .

Corollary 1a $f'(u) = -1$ for $u > u_e$ and $f'(u) > -1$ for $u < u_e$.

Lemma 2 Suppose $(u, v) = (u, f(u))$. Then following any history h that happens with positive probability, $w_h = \underline{w}$ if $u_h \in (\underline{u}, u_e]$.

Proof. Suppose $u \in (\underline{u}, u_e]$ and the first period wage satisfies $w_1 > \underline{w}$. Then, there exists $\varepsilon > 0$ such that it is still an equilibrium if the principal lowers the first period wage to $w_1 - \varepsilon$ while holding everything else unchanged. This implies

$$f(u - \varepsilon) \geq f(u) + \varepsilon.$$

If $u = u_e$, then this immediately contradicts the definition of u_e . Moreover, this also contradicts that $f'(u) > -1$ for $u < u_e$. The proof is completed by noting that $(u, v) = (u, f(u))$ implies $(u_h, v_h) = (u_h, f(u_h))$ following any history that happens with a positive probability. ■

Now define u_0 as the smallest u in which $(u, f(u))$ is obtained by requiring the agent to put in effort in period 1. The next lemma shows that f is a straight line between $(\underline{u}, f(\underline{u}))$ and $(u_0, f(u_0))$.

Lemma 3 For $u \in [\underline{u}, u_0]$,

$$f(u) = f(\underline{u}) + \frac{u - \underline{u}}{u_0 - \underline{u}}(f(u_0) - f(\underline{u})).$$

Proof. Note that f is a concave function. The statement amounts to showing that f does not have an extremal point in (\underline{u}, u_0) .

Suppose instead there exists $u \in (\underline{u}, u_0)$ such that $(u, f(u))$ is an extremal point. In particular, this implies that $(u, f(u))$ is obtained through pure strategy in period 1. Since $u < u_0$, the first period play is either $(\underline{u}, \underline{v})$ or $(\underline{w}, qy - \underline{w})$. Let the agent's (deterministic) continuation payoff in the second period be $r(u)$. Consider three possible cases: $r(u) = u_0$, $r(u) > u_0$, and $r(u) < u_0$.

If $r(u) = u$, then it means that the agent's payoff is u and it is achieved by him receiving $w = u$ every period and not exerting effort. Therefore, the principal's payoff is $qy - w = qy - u \leq \underline{v} + \underline{u} - u < \underline{v}$. This cannot be an equilibrium payoff.

Suppose $r(u) > u$. Now consider an alternative equilibrium strategy that replicates the play for $(u, f(u))$ in period 1, and uses a continuation of $(r(u) - \varepsilon, f(r(u) - \varepsilon))$, for some small enough $\varepsilon > 0$. This new strategy gives the agent a payoff of $u - \delta\varepsilon$ and the principal a payoff of $f(u) + \delta(f(r(u) - \varepsilon) - f(r(u)))$.

By the definition of f , we must have

$$f(u - \delta\varepsilon) \geq f(u) + \delta(f(r(u) - \varepsilon) - f(r(u))).$$

On the other hand, by the concavity of f , we have that

$$f(u - \delta\varepsilon) \leq f(u) + \delta(f(r(u) - \varepsilon) - f(r(u))).$$

Combining these two equations, we have

$$f(u - \delta\varepsilon) = f(u) + \delta(f(r(u) - \varepsilon) - f(r(u))),$$

but this implies the slope in $(u - \delta\varepsilon, u)$ and the slope in $(r(u) - \varepsilon, r(u))$ are the same. This is a contradiction.

If $r(u) < u$, a similar approach will also show that

$$f(u + \delta\varepsilon) = f(u) + \delta(f(r(u) + \varepsilon) - f(r(u))),$$

so the slope in $(r(u), r(u) + \varepsilon)$ and the slope in $(u, u + \delta\varepsilon)$ are the same. Once again, this contradicts that $(u, f(u))$ is an extremal point. ■

The next lemma gives the exact value of u_0 . In particular, it follows directly from

$$L(u_0) = \underline{u}.$$

Lemma 4:

$$u_0 = (1 - \delta)(\underline{w} - c) + \delta(\underline{u} + pk)$$

Proof. It is clear $L(u_0) \geq \underline{u}$, where \underline{u} is the agent's maxmin payoff. Now if $L(u_0) > \underline{u}$, we argue that there exists a PPE payoff that gives the agent the payoff of $u_0 - \varepsilon$ and the principal a payoff that lies on the line segment between $(\underline{u}, f(\underline{u}))$ and $(u_0, f(u_0))$, and this violates the definition of u_0 . In particular, let s be the slope between $(\underline{u}, f(\underline{u}))$ and $(u_0, f(u_0))$. Then by the weak concavity of f , we know that both $(L(u_0) - \varepsilon, f(L(u_0)) - s\varepsilon)$ and $(L(u_0) + k - \varepsilon, f(L(u_0) + k) - s\varepsilon)$ are PPE payoffs. And a strategy profile that pays the agent $w_1 = \underline{u}$, requires the agent to put in effort in period 1, and promises the agent $(f(L(u_0)) - s\varepsilon)$ when the output is 0 and $(f(L(u_0)) + k - s\varepsilon)$ when the output is y , will be an equilibrium. Since the agent receives a

payoff of $(u_0 - \varepsilon, f(u_0) - s\varepsilon)$ yet puts in effort in period 1, it contradicts the definition of u_0 . Therefore, $L(u_0) = \underline{u}$. ■

Lemma 5 There exists $w^* \leq \underline{u}$ such that $f(\underline{u}) = \underline{v}$ if and only if $\underline{w} \geq w^*$.

Proof. Here we make explicit that f depends on \underline{w} by using the notation $f(u, \underline{w})$. Suppose the agent's continuation payoff is u . Let $r(u|\underline{w})$ be his next period continuation payoff if in this period he receives an instantaneous payoff of \underline{u} and exerts no effort, and let $r(u|\underline{u})$ be his next period continuation payoff if in this period he receives an instantaneous payoff of \underline{u} and exerts no effort.

Step 1. We establish that $f(\underline{u}) = \underline{v}$ if and only if $(1 - \delta)(qy - \underline{w}) + \delta f(r(\underline{u}|\underline{w}), \underline{w}) < \underline{v}$.

Since $(\underline{u}, f(\underline{u}))$ is an extremal point, the first period play is pure. There are two possible instantaneous payoff profiles in the first period: $(\underline{u}, \underline{v})$ and $(w, qy - w)$ for some $w \geq \underline{w}$. By Lemma 2, $w = \underline{w}$.

If players' instantaneous payoffs are $(\underline{u}, \underline{v})$ in the first period, then $r(\underline{u}, \underline{u}) = \underline{u}$. This implies the principal's continuation payoff is \underline{v} when the agent's continuation payoff is \underline{u} . If players' instantaneous payoffs are $(\underline{w}, qy - \underline{w})$, then the principal's payoff is

$$F(\underline{w}) := (1 - \delta)(qy - \underline{w}) + \delta f(r(\underline{u}|\underline{w}), \underline{w}).$$

Clearly $f(\underline{u}, \underline{w}) = \max\{F(\underline{w}), \underline{v}\}$.

Step 2. Now we show that if $f(\underline{u}, \underline{w}) > \underline{v}$ when minimum wage is \underline{w} , then $f(\underline{u}, \underline{w} - \varepsilon) > \underline{v}$ for $\varepsilon > 0$ must hold when minimum wage is $\underline{w} - \varepsilon$.

Suppose $(1 - \delta)(qy - \underline{w}) + \delta f(r(\underline{u}|\underline{w})) > \underline{v}$. Notice that f is weakly decreasing in \underline{w} . Also notice that

$$\begin{aligned} (1 - \delta)\underline{w} + \delta r(\underline{u}|\underline{w}) &= (1 - \delta)(\underline{w} - \varepsilon) + \delta r(\underline{u}|\underline{w} - \varepsilon) = \underline{u} \\ r(\underline{u}|\underline{w} - \varepsilon) &= r(\underline{u}|\underline{w}) + \frac{1 - \delta}{\delta}\varepsilon. \end{aligned} \tag{12}$$

Then

$$\begin{aligned} &(1 - \delta)(qy - (\underline{w} - \varepsilon)) + \delta f(r(\underline{u}|\underline{w} - \varepsilon), \underline{w} - \varepsilon) \\ &\geq (1 - \delta)(qy - (\underline{w} - \varepsilon)) + \delta f\left(r(\underline{u}|\underline{w}) + \frac{1 - \delta}{\delta}\varepsilon, \underline{w}\right) \\ &\geq (1 - \delta)(qy) + \delta f(r(\underline{u}|\underline{w}), \underline{w}) > \underline{v}. \end{aligned}$$

The first inequality follows (12) and the fact that f is weakly decreasing in \underline{w} . The second inequality follows the fact that $f_1(r(\underline{u}|\underline{w}), \underline{w}) \geq -1$. This completes the proof. ■

In the following two lemmas, we pin down the value of u_e .

Lemma 6a If $py - \frac{[1-\delta(1-p)]c}{\delta(p-q)} \geq \underline{v} + \underline{w}$, then

$$u_e = L(u_e) = (\underline{w} - c) + \frac{\delta pk}{(1 - \delta)}.$$

Proof. Note that $u = (1 - \delta)(\underline{w} - c) + \delta[pL(u) + (1 - p)(L(u) + k)]$ for $u \geq u_0$, which can be rewritten as

$$L(u) = \frac{u - (1 - \delta)(\underline{w} - c) - \delta(1 - p)k}{\delta}.$$

Define u_r by $L(u_r) = u_r$. Then

$$\begin{aligned} u_r &= \frac{u_r - (1 - \delta)(\underline{w} - c) - \delta(1 - p)k}{\delta} \\ u_r &= \underline{w} - c + \frac{\delta pk}{1 - \delta}. \end{aligned}$$

The assumption of the lemma can be rewritten as $u_r + k \leq py - c - \underline{v}$. This implies it is an equilibrium in which the agent is paid a contractable wage \underline{w} and a bonus $k = (1 - \delta)c / [\delta(p - q)]$ (in form of a higher wage in the following period) if and only if the outcome is good and exerts effort every period. In other words, $f(u_r) = py - c - u_r$. By definition of u_e , $u_e \leq u_r$ and $f(u_e) = py - c - u_e$. Since $L'(u) = 1/\delta$, if $u_e < u_r$, then $L(u_e) < u_e$. By definition of u_e , efficiency is not achieved at $L(u_e)$.

On the other hand, for efficiency to hold at u_e ,

$$(1 - \delta)(py - c) + \delta[(1 - p)[L(u_e) + f(L(u_e))] + p[L(u_e + k) + f(L(u_e + k))] = py - c$$

This implies that

$$[L(u_e) + f(L(u_e))] = [L(u_e + k) + f(L(u_e + k))] = py - c$$

This contradicts that efficiency is not achieved at $L(u_e)$. ■

Lemma 6b: Suppose $py - \frac{[1-\delta(1-p)]c}{\delta(p-q)} < \underline{v} + \underline{w}$, then

$$u_e = (1 - \delta)(\underline{w} - c) + \delta[u_{\max} - (1 - p)k].$$

Proof. Note that $L(u_e) + k \leq u_{\max}$ must hold. The assumption of the lemma implies that $L(u_r) + k > u_{\max}$. Since $L(u_e) + k \leq u_{\max}$, it follows that $u_e < u_r$. This implies $L(u_e) < u_e$.

Suppose $L(u_e) + k < u_{\max}$. Then consider the equilibrium profile $(u_e + \varepsilon, f(u_e + \varepsilon))$. We

know that the slope of f for $u > u_e$ is -1 . This implies that

$$f(u_e + \varepsilon) = f(u_e) - \varepsilon.$$

On the other hand,

$$f(u_e + \varepsilon) \geq (1 - \delta)(py - \underline{w}) + \delta((1 - p)f(L(u_e + \varepsilon)) + pf(L(u_e + \varepsilon) + k)).$$

Now for small enough ε , we have $L(u_e + \varepsilon) < u_e$. Now since $f'(u) > -1$ for $u < u_e$, the above implies that $f'(u_e + \varepsilon) > -1$ for small enough ε . Therefore, we have

$$f(u_e + \varepsilon) > f(u_e) - \varepsilon,$$

and this contradicts the definition of u_e . The lemma follows immediately $L(u_e) + k = u_{\max}$. ■

Lemma 7: For $u \in [u_0, u_e]$,

$$f(u) = (1 - \delta)(py - \underline{w}) + \delta[pf(L(u) + k) + (1 - p)f(L(u))]. \quad (13)$$

Proof. This follows in two steps. The first step shows that for $u \in [u_0, u_e]$, $(u, f(u))$ can be obtained by an equilibrium profile in which the first period play requires effort. Now suppose the contrary. Let u_l be the largest point such that $(u_l, f(u_l))$ lies on the line given by $(\underline{u}, f(\underline{u}))$ and $(u_0, f(u_0))$.

Suppose $u \in [u_l, u_e]$. There are two cases to consider. In the first case, $(u, f(u))$ can be reached by a pure play in period 1. In this case, the first period play payoff is given by either $(\underline{u}, \underline{v})$ or $(\underline{w}, qy - \underline{w})$. In either case, $(u, f(u))$ is a convex combination between the instantaneous payoff and some continuation payoff, which we denote by $(u_r, f(u_r))$. Due to concavity of f , $(u, f(u))$ is dominated by a mixing between $(u_0, f(u_0))$ and $(u_r, f(u_r))$.

In the second case, the $(u, f(u))$ is reached through a mix. Since $(u, f(u))$ lives in a two dimension space and f is concave, we may assume

$$(u, f(u)) = p_1((u_1, f(u_1))) + (1 - p_1)((u_2, f(u_2)))$$

for some p_1, u_1 , and u_2 , where $(u_1, f(u_1))$ and $(u_2, f(u_2))$ are reached through pure play in period 1. If $(u_i, f(u_i))$, $i = 1, 2$ has first period play that doesn't require effort, the previous paragraph implies that $u_i \notin [u_l, u_e]$. But then the concavity of f implies that if $(u, f(u))$ can be obtained by a linear combination with the agent's payoff $u_i \notin [u_l, u_e]$, then it can also be obtained by a linear combination of points with the agent's payoff belonging to $[u_l, u_e]$.

To finish the first step, if $u \in (u_0, u_l)$, then $(u, f(u))$ can be achieved by mixing between $(u_0, f(u_0))$ and $(u_l, f(u_l))$. And since both $(u_0, f(u_0))$ and $(u_l, f(u_l))$ can be achieved by requiring

effort in period 1, so can $(u, f(u))$.

In the second step, we note that the equation follows because to achieve the maximum payoff for the principal, a) the continuation payoff must lie on the Pareto frontier, and b) the distance of payoff between the good and bad outcomes for the agent needs not to exceed k by the concavity of f . ■