Performance and Turnover in a Stochastic Partnership

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Abstract

This paper characterizes the welfare-maximizing equilibrium performance and duration of stochastic partnerships, in an economy in which partners choose each period costly observable efforts, voluntary wages, and whether to leave the relationship to be re-matched. Individuals' lives in this economy tend to transition between a few qualitatively distinct phases: "dating" at birth or between relationships; "honeymoon"; "hard times"; "good times"; and "golden years", from which partners are parted only by death. Given an exogenous stochastic process, higher states are associated with higher stage-game and continuation payoffs, as well as longer-lasting relationships.

1 Introduction

Players in an ongoing interaction often face uncertainty regarding the fundamentals of their relationship. For example, an employer may be unsure about whether his worker will have an incentive in the future to accept an outside offer. Or, firms engaged in a joint venture may be unsure about future payoffs within their partnership. Such uncertainty can make it difficult to sign complete formal contracts, especially if what might change in the relationship is difficult to communicate to an outside party. At the same time, a long-lasting (or "stable") relationship is crucial for the effective provision of informal incentives. If shocks to the

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productivity of a partnership may cause it to end or be less productive in the near future, players will have less incentive to work today, reducing relational gains and potentially hastening the partnership's demise in a vicious cycle.

Partnerships in which players face an uncertain future can generate rich dynamics. For example, one of General Motors' most important joint ventures is Shanghai GM, a 50-50 partnership with Shanghai Automotive Industry Corp (SAIC) formed in 1997 that has grown into a fully integrated operation that moves over one million cars per year. This venture allows GM to build cars in China, while providing SAIC with valuable expertise. In fact, GM has aggressively transferred vehicle development know-how to its Chinese partner, as recently as 2008 with the complete re-design of the Buick Regal for the Chinese market, despite some mimicking of its past designs in SAIC vehicles. Yet, when announcing plans in 2007 to build a hybrid-engine research center in Shanghai, GM chose not to work with SAIC, even as the Chinese government announced seemingly strict rules requiring the manufacture of hybrid engines on Chinese soil (Bradsher (2007)).

As in many uncertain relationships, this partnership's future is subject to a variety of risks, some within the players' control (will GM continue to refuse to share valuable intellectual property, will SAIC continue to copy-cat GM designs?) and others less so. Indeed, the future of GM Shanghai may well hinge on developments in another stochastic relationship, that between the Chinese government and foreign auto-makers. For example, should China stop foreign firms from exporting earnings, or allow them to build hybrid cars without a Chinese partner, GM would have much less incentive to share its technology.

This paper develops a theory of *endogenous stability and performance* in a perfect-information model of stochastic partnerships. Each period, two partners simultaneously decide how much effort to exert after observing a payoff-relevant state. "Effort" can be interpreted broadly, e.g. to include costly relationship-specific investments. After observing efforts, the partners then simultaneously decide whether to quit the relationship. The partnership ends if either player quits, in which case each player receives an outside option. Also, players can make voluntary wage transfers at any time during the game, although I show that it is without loss to restrict attention to "retention bonuses" paid after and only if both players choose not to quit. (See Figure 1 in Section 2.)

Incremental payoffs from higher effort satisfy an increasing differences property in a state variable that follows a controlled stochastic process, i.e. the current state may depend on past states and past efforts. The main restriction on this process is that it is persistent, i.e. higher past states make higher future states more likely in the sense of first-order stochastic dominance, but no substantive restrictions are placed (at first) on how efforts control the stochastic process. This allows for a rich set of potential applications from labor to macroeconomics and organizational economics, in which greater effort grows, depletes, or has a non-monotone effect on a payoff-relevant relational stock. For example, in a labor context, one could interpret the worker's (multi-dimensional) effort as including hours worked as well as investments in firm-specific human capital. The assumptions are sufficiently weak that the existing literature on comparative statics in stochastic games does not apply. (See the literature discussion below.)

This paper derives a subgame-perfect equilibrium (SPE) that maximizes players' joint welfare among all SPE. Joint payoff in this "optimal equilibrium" is non-decreasing in the state, but higher states need *not* be associated with higher joint stage-game payoff or higher joint continuation payoff. Consequently, players in higher states may or may not exert more effort, may or may not exit with lower probability, etc.

Such partnerships are then embedded within a "partnership economy" with anonymous re-matching after partnership dissolution. If some player's partnership ends at time t in this economy, whether because he left, his partner left, or his partner died, he is automatically re-matched with a new partner to begin at time $t + 1$. This new partnership is a "fresh start", in the sense that (i) the stochastic processes driving stage-game payoffs are iid across partnerships and (ii) players know nothing about their current partner's history before their partnership began, including his age, number of past partnerships, etc.[1] Expected payoffs in a new partnership generate outside options for each player should his current partnership end. The analysis endogenizes the maximal joint outside option that can be supported in any equilibrium of the overall partnership economy. Further, given an exogenous inflow and

[1] If historical variables such as age could be observed, then economy-wide welfare might be enhanced in "old-maid equilibria" in which players who are not newly-born are shunned, since no one will ever leave a relationship in such an equilibrium.

outflow of births and deaths, I characterize the steady-state distribution of histories among active partnerships in this welfare-maximizing equilibrium.

At birth or whenever unmatched in the steady state of this economy, a player will typically sample several partners before settling into a new relationship, for two related but distinct reasons. First, if the initial state of a partnership is informative of future payoff possibilities, each player wants to sample extensively in order to find a "good fit". More interestingly, players will act *as if* initial states matter greatly even when they do not. For instance, consider the most extreme case in which all partnerships are payoff identical at first, but in which each partnership receives a random public label from zero to one. In the welfare-maximizing equilibrium, players will exert no effort and immediately leave all partnerships in which the label is not sufficiently close to one. Delay in this "dating" process emerges as an endogenous solution to the incentive problem created by players' ability to leave for a new match, similar to the well-known "incubation period" in non-stochastic repeated games with re-matching. (See the literature discussion below.)

Since players immediately leave all but the best matches, there is a "honeymoon effect" to partnership formation. Namely, partnerships that persist at all are likely to last a relatively long time and to be highly productive at first. Exit is triggered when the state of a partnership falls below a threshold-surface in the state-space. Consequently, partnerships that have lasted a long time tend to be those that have received mostly positive shocks that made the partnership more stable. This *survivorship bias* is consistent with a broad empirical finding that, from employment (Topel and Ward (1992)) to marriage (Stevenson and Wolfers (2007)) and organizational survival (Levinthal (1991)), partnerships that have lasted a long time are less likely to end in the near future.

There is a range of states ("hard times") in which partners exert little or no effort in the optimal equilibrium but elect to remain together despite this failure to cooperate. Players endure such hard times, rather than exiting, because of the *option value* associated with waiting to exit. However, this option value does not only arise as usual from exogenous variation in the productivity of the partnership itself. The option to exit later becomes more valuable, in equilibrium, because of the endogenous variability of players' behavior.

Finally, more comparative statics are available under the additional assumption that players' efforts do not control future payoff possibilities. In this case, partnerships in higher states will enjoy (weakly) higher stage-game payoffs as well as higher continuation payoffs, and persist longer into the future.

The rest of the paper is organized as follows. The introduction continues with discussion of some related literature. Section 2 describes the model while Section 3 provides a simple example. Section 4 characterizes the joint-welfare maximizing SPE in a partnership for any outside options. Section 5 then characterizes the social-welfare maximizing equilibrium of the overall partnership economy, thereby endogenizing each player's outside option. Finally, Section 6 develops more comparative statics in the special case of an exogenous stochastic process, while Section 7 provides concluding remarks. Most proofs are in an Appendix.

Related literature.

This paper combines elements of Jovanovic (1979a) and Levin (2003) in a rich stochastic framework with two-sided incentives. Jovanovic (1979a) considers a model in which a worker learns over time about the productivity of the match with his present firm and quits as soon as he becomes sufficiently pessimistic about the match. Consequently, workers who have remained longer at the same firm are less likely to leave and more likely to be more productive.[2] The key difference here is that partners face a two-sided incentives problem as well as a learning problem. Whereas the worker in Jovanovic always enjoys the full gains from his current match, players here must work to enjoy those gains and choose how to distribute them through voluntary wages.

Levin (2003) characterizes optimal "relational contracts" in a principal-agent context in which the agent's cost of effort is iid. Unlike Levin (2003), this paper allows for two-sided incentives and non-iid stage-game payoffs. (However, Levin's analysis is *not* less general, as he allows for incomplete information and imperfect monitoring of effort.) Also, a key step in the analysis here is to show that performance *inside* the partnership decreases with the attractiveness of players' outside options. This extends a well-known finding of the relational contracts literature (see e.g. MacLeod and Malcomson (1989) and Baker, Gibbons, and

[2] Also closely related is Jovanovic (1979b), in which similar effects arise as workers who choose to remain in their current job make firm-specific investments to improve the future performance of the match. See also Pissarides (1994) for a related model of on-the-job search.

Murphy (1994)) to a richer stochastic setting.

This paper also adds to the "dynamic games" literature in which a payoff-relevant state follows a known stochastic process.[3] For example, a key insight in Haltiwanger and Harrington (1991) and Bagwell and Staiger (1997)'s models of collusion and the business cycle, that collusion thrives at those times when the *future* state is most likely to be conducive to collusion, is helpful for interpreting this paper's results as well. However, the focus here is on how players' ability to dissolve their partnership interacts with their incentive to exert costly effort. Also, by allowing for any persistent stochastic process, my analysis encompasses both the iid case (as in Rotemberg and Saloner (1986), Ramey and Watson (1997)) and the "positively autocorrelated" case (as in Bagwell and Staiger (1997)), among others.

Like this paper, Roth (1996) shows how to construct welfare-maximizing equilibria in a dynamic partnership, using an algorithm in the spirit of Abreu, Pearce and Stacchetti (1990). Indeed, Roth's model can be viewed as a special case of mine in which, among other things, payoffs are symmetric, the state is one-dimensional and follows a simple random walk, and there is no feedback of effort on future states. Also, this paper differs by endogenizing the players' outside options via re-matching.

Repeated games with re-matching opportunities have been studied by several authors, including Kranton (1996), Datta (1996) and recently Eeckout (2006).[4] A key finding is that, in welfare-maximizing equilibria of the overall partnership economy, partners fail to realize all potential equilibrium gains in their individual partnerships, instead enduring an "incubation period" with low efforts and low payoffs before transitioning to a maximally productive phase. The analysis here will show that such results are not robust. Under mild conditions on the stochastic process driving payoffs (see Theorem 2 and Claim 2), welfare-maximizing equilibria of the overall partnership economy dictate renegotiation-proof play within each partnership in which players "date" several partners rather than "starting small" and enduring an incubation period with just one partner. Intuitively, in a stochastic setting in which not all partnerships are equal, the dating process allows players to match

[3] A growing and less closely related literature considers dynamic games in the presence of imperfect information, e.g. Athey and Bagwell (2001) and Horner and Jamison (2007).

[4] Separately, in an incomplete information setting, Ghosh and Ray (1996) and Watson (1999) provide a signaling rationale for "starting small".

with partners who are a relatively good initial fit.

Recently, Chassang (2009) and Bonatti and Horner (2009) have developed other theories of cooperation dynamics. Namely, Chassang (2009) shows how players "build routines" in repeated games with incomplete information about payoffs, while Bonatti and Horner (2009) develop a theory of dynamic public good provision given unobserved efforts and uncertainty about the quality of the public good. In each of these papers, the underlying environment does not change over time. This paper highlights dynamics that arise when payoffs are stochastic, while abstracting from (important) issues of incomplete information and imperfect monitoring.

More tangentially related is the existing literature on "stochastic games", especially those papers such as Amir (1996) and Curtat (1996) in which sufficient monotone structure is imposed to generate comparative statics. However, most of these papers focus on equilibria in Markov strategies, often proving uniqueness of such equilibria, whereas I consider subgame-perfect equilibria (SPE) and focus on the SPE that maximizes joint welfare among all SPE. Further, this literature imposes stronger assumptions than are needed here, in large part because they prove stronger results (such as uniqueness).

Lastly, although players have the option to exit, the literature on so-called "option games" is unrelated. In an option game, players' payoffs depend upon who exercises a real option (e.g. exiting a market) and when they do so, and papers in this literature tend to focus on issues of strategic pre-emption or delay that arise when players prefer to be the first or last to exercise their option. See e.g. Grenadier (2002) and Chassang (2007). By contrast, my focus is to endogenize the payoffs that can be realized *prior* to exit.

2 Model

Two (potentially asymmetric) players in a partnership each seek to maximize the expected present value of their stream of payoffs, given per-period discount factor $\delta < 1$. All assumptions presented here apply throughout the analysis. Those meriting further discussion are highlighted and numbered. (Additional assumptions will be explicitly stated when needed.)

Notational shorthand. To improve clarity and shorten equations, I have adopted several notational conventions throughout the paper. First, random variables are capitalized while realizations are in lower case, e.g. $x_t \in \text{supp}(X_t)$. Second, variables specific to a player and time have two subscripts, e.g. e_{it}. Vectors of such variables for all players and *all times no later than* t are bolded with one subscript, e.g. $\mathbf{e}_t = (e_0, ..., e_t)$, while those for all players *at one time* t are unbolded with one subscript, e.g. $\pi_t(e_t; x_t) = (\pi_{it}(e_{it}, e_{jt}; x_t), \pi_{jt}(e_{it}, e_{jt}; x_t))$. Finally, *sums* are denoted by a summation subscript, e.g. $\pi_{\Sigma t}(e_t; x_t) = \sum_i \pi_{it}(e_t; x_t)$.

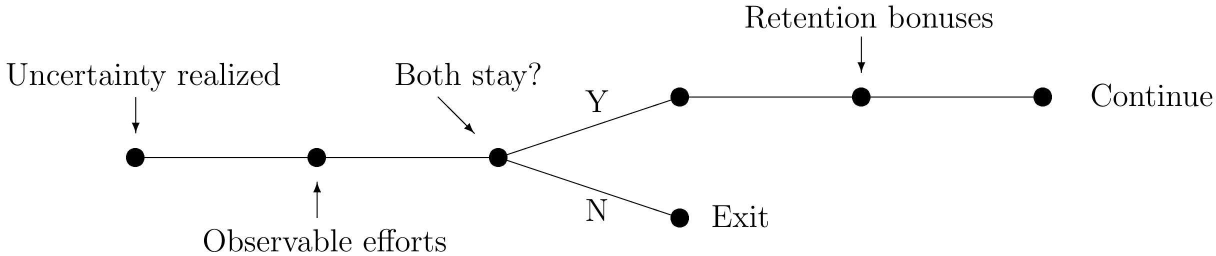

Figure 1: Timing of the partnership stage-game in period $t = 0, 1, 2, ...$

Partnership stage-game. Each period $t = 0, 1, 2, ...$ proceeds as follows until the partnership ends (see Figure 1). First, a payoff-relevant state $x_t \in \mathcal{X}_t$ is realized and publicly observed. $(\mathcal{X}_t, \succeq)$ is a partially ordered set. Second, each player i simultaneously decides what effort $e_{it} \in \mathcal{E}_{it}$ to exert, where efforts may control the stochastic process $(X_t : t \geq 0)$. Efforts are then publicly observed and each player i receives "inside payoff" $\pi_{it}(e_t; x_t)$. $(\mathcal{E}_{it}, \succeq)$ is a partially ordered, complete topological space having minimal element "0".[5]

Assumption 1 (Inside payoffs)**.** $\pi_{it}(e_t; x_t)$ is weakly decreasing in e_{it}, weakly increasing in e_{jt}, and continuous in e_t, $\pi_{it}(0, 0; x_t) = 0$ for all x_t, and $\pi_{\Sigma t}(e_t; x_t)$ is uniformly bounded above.

Assumption 2 (Increasing differences)**.** π_{it} satisfies weakly increasing differences in $(e_t; x_t)$. That is, $e_t^H \succeq e_t^L$ and $x_t^H \succeq x_t^L$ implies $\pi_{it}(e_t^H; x_t^H) - \pi_{it}(e_t^L; x_t^H) \geq \pi_{it}(e_t^H; x_t^L) - \pi_{it}(e_t^L; x_t^L)$.

[5]The effort-set $\mathcal{E}_{it}$ can be viewed as the set of all mixtures over some underlying set of pure efforts, endowed with the first-order stochastic dominance ordering inherited from the partial order on that underlying set. The assumption here of observable efforts then corresponds to an assumption that effort *mixtures* are observed. If mixtures cannot be observed, this paper's results can be viewed as characterizing welfare-maximizing *pure-strategy* SPE.

Assumption 3 (State)**.** The distribution of X_t depends only on $(t, x_{t-1}, \mathbf{e}_{t-1})$.

Third, each player i simultaneously decides whether to stay or quit the partnership. The partnership ends if either exits *and*, should both stay, with exogenous probability $\lambda \in [0, 1]$. (λ corresponds to the rate of "death" in the application of Section 5.) If so, each player i receives an outside option having present value $v_i \geq 0$ and nothing thereafter.[6] Otherwise, the partnership remains active in period $t + 1$.

Throughout the game, players can make voluntary wage transfers to one another at any time. However, it is without loss to restrict attention to SPE in which only "retention bonuses" are paid each period, after and only if both players decide to stay (Lemma 1). For simplicity, then, I will proceed as if this is the only time at which players can pay wages. This is mainly for expositional clarity, as the analysis can be adapted to settings with other wage timing. (See the related discussion in Levin (2003).)

Solution concept. The solution concept is subgame-perfect equilibrium (SPE), with special focus on SPE that maximize players' joint welfare among all SPE.

Stochastic process. The stochastic process $(X_t : t \geq 0)$ satisfies a "persistence" property, that future states are more likely to be higher when the current states is higher. Two definitions are needed to make this precise.

Definition 1 (Increasing subset)**.** Let $(\mathcal{Z}, \geq)$ be any partially-ordered set. $\mathcal{Y} \subset \mathcal{Z}$ is an "increasing subset of $\mathcal{Z}$" if $a_1 \in \mathcal{Y}$ and $a_2 \geq a_1 \in \mathcal{Z}$ implies $a_2 \in \mathcal{Y}$.

Definition 2 (Generalized first-order stochastic dominance[7])**.** Let A_1, A_2 be random variables with support in partially ordered set $(\mathcal{Z}, \geq)$. A_1 "first-order stochastic dominates"

[6] The analysis and basic results extend to settings in which players' outside options $V_t = (V_{it}, V_{jt})$ are random, as long as they are not controlled by players' efforts. (In particular, V_t may be correlated with X_t, but only if X_t follows an exogenous stochastic process as in Section 6.) Extending the analysis to allow for endogenous outside options, as when players may search for their next partner while still matched, is an important direction for future research.

[7] When $\mathcal{Z} = \mathbf{R}$, this condition reduces to the familiar requirement that $\Pr(A_1 \geq z) \geq \Pr(A_2 \geq z)$ for all $z \in \mathbf{R}$. There is more than one natural way to generalize FOSD to multi-dimensional settings, some more restrictive than the notion used here. See e.g. Stoyan and Daley (1983).

(FOSD) A_2 if $\Pr(A_1 \in \mathcal{Y}) \geq \Pr(A_2 \in \mathcal{Y})$ for all increasing subsets $\mathcal{Y} \subset \mathcal{Z}$.

Assumption 4 (Persistence)**.** $x'_t \succeq x_t$ implies $X_{t+1}|(x'_t, \mathbf{e}_t)$ FOSD $X_{t+1}|(x_t, \mathbf{e}_t)$ for all $\mathbf{e}_t$.

Definition 3 (Cost of effort)**.** Let $c_{it}(e_t; x_t) = \sup_{\tilde{e}_{it}} (\pi_{it}(\tilde{e}_{it}, e_{jt}; x_t) - \pi_{it}(e_t; x_t))$ denote each player's "cost of effort e_{it}" when player j exerts effort e_{jt} at time t.

Discussion of the model: By Assumption 1, each player has a weakly dominant strategy to exert zero effort in each effort stage-game, so $c_t(e_t; x_t) = \pi_{it}(0, e_{jt}; x_t) - \pi_{it}(e_t; x_t)$. By Assumption 2, $x'_t \succ x_t$ implies

$$\pi_{it}(e_t; x'_t) \geq \pi_{it}(e_t; x_t) \text{ for all } e_t \tag{1}$$

$$c_t(e_t; x'_t) \leq c_t(e_t; x_t) \text{ for all } e_t \tag{2}$$

for all e_t. (Increasing differences implies (1) when we set $e_t^H = e_t$ and $e_t^L = (0, 0)$ and implies (2) when we set $e_t^H = e_t$ and $e_t^L = (0, e_{jt})$.)

Assumption 3 requires that inside productivity and outside options be unrelated. Interactions between inside and outside payoffs can arise quite naturally, for at least two reasons. First, outside option values may be correlated with the partnership state for exogenous reasons, if there is some common factor driving both. For example, a player's outside option might be to start his own business in the same line of work as that pursued by the partnership. Second, "search" activities that enhance a player's outside option may also enhance or detract from the future productivity of the partnership.

Assumption 4 states that the partnership is weakly more likely to transition to a "better state" tomorrow from a better state today, holding fixed the history of players' efforts. The fact that no assumptions are made on how efforts impact future states allows for great flexibility, e.g. the model can accommodate settings in which effort grows, depletes, or has a non-monotone effect on a payoff-relevant stock. On the other hand, Assumption 4 does rule out a variety of potential applications in which payoffs are stochastic but not persistent. For instance, suppose that $\mathcal{X}_t = \{\text{low}, \text{high}\}$ for all t as in Bagwell and Staiger (1997). Assumption 4 fails in the case of negative serial auto-correlation.

Here are some simple examples of state processes $(X_t : t \geq 0)$ satisfying Assumption 4. In each case, $\mathcal{X}_t \subset \mathbf{R}^K$. Examples (A,B) are exogenous Markov processes, (C) is a non-Markov exogenous process, (D) is a non-trivially controlled process.

(A) X_t are iid.

(B) $g(X_t)$ is a random walk where $g : \mathbf{R}^K \to \mathbf{R}^K$ is any non-decreasing function relative to the usual product order on $\mathbf{R}^K$.

(C) $(X_s : s \in [0,t])$ is a sequence of publicly observed estimates of K unobserved parameters, e.g. unknown productivity of the match à la Jovanovic (1979a).

(D) $X_t = Z_t(X_{t-1} + \sum_i e_{it})$, where $(Z_t : t \geq 0)$ is an exogenous stochastic process as in any of the previous examples. For instance, X_t and $Z_t - 1$ could be the value and growth rate, respectively, of accumulated capital in a joint venture with new capital investments e_t at time t.

3 Example: Dynamic Prisoners' Dilemma

This section provides a simple example that serves to fix ideas and highlight some aspects of the more general analysis to come.

	Work	Shirk
Work	$1, 1$	$-1 - c_t, 1 + c_t$
Shirk	$1 + c_t, -1 - c_t$	$0, 0$

Figure 2: Stage-game payoffs at time t, while the partnership persists.

Each stage-game has stage-game payoffs as shown in Figure 2.[8] "Effort cost" $c_t > 0$, and $\log(C_t)$ evolves according to a known and exogenous random walk with iid motion that is atomless on support that contains $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Should the partnership dissolve, each player gets the same outside option having value $v \geq 0$. All assumptions of Section 2 are satisfied when we define the state $x_t = -c_t$.

[8] Such payoffs arise naturally in a context in which players bear all of the cost of their own effort but share equally the return to that effort. Suppose that each player generates a return equal to his cost when working alone, but generates an excess return of one when working together with the other player. The payoffs of Figure 2 then arise when the cost and return of individual effort is $2(1 + c_t)$.

If $c_t = c_0$ for all t, then this is a standard repeated Prisoners' Dilemma. However, as I will show, welfare-maximizing equilibrium behavior changes dramatically once we allow $(C_t : t \geq 0)$ to follow a non-trivial stochastic process. Results to be presented later in the paper imply that a "threshold equilibrium" maximizes joint welfare among all SPE in this setting.[9] The optimal work threshold $c^{W*} = \frac{\delta}{1-\delta}$ in the standard repeated-game setting, but equals only $\frac{\delta}{2(1-\delta)}$ when $\log(C_t)$ follows any symmetric random walk with atomless transitions.

Definition 4 (Threshold equilibrium)**.** A SPE is a "threshold equilibrium" if there exists a "work threshold" c^W and "exit threshold" $c^E \geq c^W$ such that, on the equilibrium path, both players (a) work and stay if $c_t \leq c^W$, (b) shirk and stay if $c^W < c_t \leq c^E$, and (c) shirk and quit if $c_t > c^E$; off the equilibrium path, both players shirk and quit in all states.

Players' behavior in this "optimal SPE" is summarized by Figure 3. Since the outside option is fixed, the state can be viewed as moving up and down a vertical slice of this figure. When the region labeled "EXIT" is reached, both players shirk and quit. Until then, both players work and stay when in the region labeled "WORK" while both shirk and stay when in the region labeled "SHIRK". The partnership is "doomed" whenever the players' outside payoff is close enough to their continuation payoff in a productive partnership, in the sense that the players shirk and exit immediately in every state in every SPE.

Intuitive derivation of optimal thresholds. *Exit threshold.* Consider any threshold equilibrium with work threshold c^W. In such an equilibrium, the partnership can be viewed as a jointly owned asset that, prior to liquidation, generates payoff two when $c_t \leq c^W$ and zero when $c_t > c^W$, and that is worth $2v$ upon liquidation. Since $(\log(C_t) : t \geq 0)$ is a random walk, the optimal time to exercise the option to liquidate is when the state first exceeds $\alpha^* c^W$, for some $\alpha^* > 1$ that does not depend on c^W. Assuming that players can

[9] Since $(C_t : t \geq 0)$ is an exogenous stochastic process, joint stage-game payoff and joint continuation payoff are non-increasing in c_t in a joint-welfare maximizing SPE by Theorem 3. Since joint payoff is two if both players work and zero otherwise, monotonicity of joint stage-game payoff implies that this SPE has a work threshold (that may vary with history), while monotonicity of joint continuation payoff implies that this SPE has an exit threshold (that may also vary with history). Finally, since $(\log(C_t) : t \geq 0)$ is a random walk, it is straightforward to show that the optimal work and exit thresholds do not depend on history.

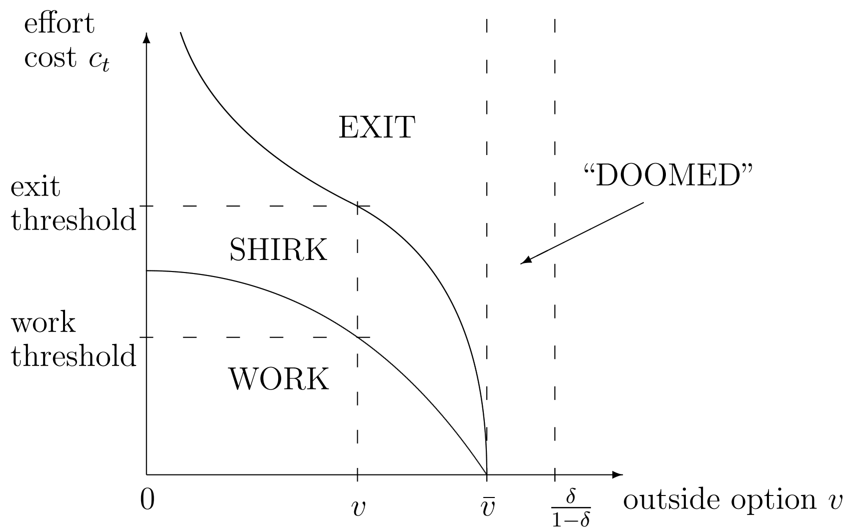

Figure 3: Summary of behavior in the optimal SPE.

support work threshold c^W in SPE, they can also support the optimal exit threshold given c^W, which is $c^E = \alpha^* c^W$. (Intuitively, equilibrium exit can be socially efficient because the players have no conflict of interest when it comes to the timing of exit, since each player gets the same outside option v.[10])

Work threshold. Threshold equilibrium payoffs are clearly increasing in the work threshold. What then is the maximal work threshold c^W that can be supported in SPE, assuming an optimal exit threshold $c^E = \alpha^* c^W$? Each player can be induced to work at time $t = 0$ (or similarly at any time) only if continuing inside the partnership is more valuable than $c_0 + v$, since otherwise he would prefer to shirk and quit when the other player works. Given work threshold c^W and exit threshold $c^E = \alpha^* c^W$, each player's continuation payoff $\Pi_{i0}^{cont}(c_0; c^W)$ when the current state is c_0 takes the form

$$\sum_{t \geq 1} \delta^t \left(\Pr\left(\max_{1 \leq s < t} C_s \leq \alpha^* c^W, C_t \leq c^W | c_0 \right) + v \Pr\left(\max_{1 \leq s < t} C_s \leq \alpha^* c^W, C_t > \alpha^* c^W | c_0 \right) \right) \quad (3)$$

Π_{i0}^{cont} is weakly decreasing in c_0 by Theorem 3. Thus, to prove that both players have sufficient incentive to work when $c_0 \leq c^W$, it suffices to check that they do so when $c_0 = c^W$. Observe next that $\Pi_{i0}^{cont}(c^W; c^W)$ does not depend on c^W. (Intuitively, continuation payoffs depend on how frequently the state will be below c^W and how quickly it will first exceed

[10]In the more general analysis to come with asymmetric players, the possibility of wage transfers is essential to align the players' incentives regarding the timing of exit. Optimal wage transfers are zero here.

$\alpha^* c^W$. Conditional on $c_0 = c^W$, this translates as how frequently $\log(C_t)$ will be lower than $\log(c_0)$ and how quickly it will first exceed $\log(c_0)$ by more than $\log(\alpha^*)$, neither of which depends on c^W by the random walk assumption.) Consequently, the maximal and hence optimal SPE work threshold is simply $c^{W*} = \Pi_{i0}^{cont}(1;1) - v$.

Monotonicity of thresholds. Both the optimal work threshold c^{W*} and the optimal ratio α^* of the exit and work thresholds are decreasing in the players' outside option v. Intuitively, α^* decreases in v because staying in a temporarily unproductive relationship becomes more costly as v increases, while c^{W*} decreases in v because a more valuable outside option combined with the prospect of quicker exit decreases the future value that can be credibly created within the partnership.

Cooperation in fewer states. In an unchanging version of the game, $c_t = c_0$ for all t, SPE exist in which players both work and stay forever as long as $c_0 \leq \frac{\delta}{1-\delta} - v$. By contrast, the optimal work threshold $c^{W*} < \frac{\delta}{1-\delta} - v$ given a changing state. The following example illustrates this basic result, which holds much more generally, whenever $(\log(C_t) : t \geq 0)$ follows a stochastic process in which all states communicate.

Example 1. Suppose that $v = 0$ and that $\log(C_{t+1}) - \log(C_t) \sim U[-\varepsilon, \varepsilon]$ for some $\varepsilon \geq 0$. Let $c^{W*}(\varepsilon)$ be the optimal work threshold as a function of ε.

Claim 1. *In this example,* $c^{W*}(0) = \frac{\delta}{1-\delta}$ *while* $c^{W*}(\varepsilon) = \frac{\delta}{2(1-\delta)}$ *for all* $\varepsilon > 0$.

Proof. As discussed earlier in the intuitive derivation of c^{W*}, a property of the *optimal* work threshold is that, conditional on $c_0 = c^{W*}$, each player must expect continuation payoff of *exactly* $c^{W*} + v$. Since $v = 0$, players never exit in the optimal SPE. Thus, each player's continuation payoff takes the relatively simple form

$$\sum_{t \geq 1} \delta^t \Pr\left(C_t \leq c_0 | c_0 = c^{W*}\right) = \frac{1}{2} * \frac{\delta}{1-\delta} \tag{4}$$

for all $\varepsilon > 0$, since $\Pr\left(C_t \leq c_0 | c_0 = c^{W*}\right) = \frac{1}{2}$ for all t by the symmetry assumption. □

Discussion of Claim 1. Equilibrium cooperation is impossible whenever the players' cost of effort exceeds c^{W*}. Thus, conditional on $c_t = c^{W*}$, players anticipate that they will be able to cooperate at most half of the time in each future period. This shrinks by half the future value of the relationship, relative to an unchanging setting in which future cooperation can be credibly promised in every period, regardless of the speed at which the players' cost of effort changes over time. Note: The set of SPE payoffs here is discontinuous in ε at $\varepsilon = 0$. However, this discontinuity is itself non-robust to other changes in the game environment. In particular, if the stage-game is modified to accommodate a continuum of effort-levels, then the maximal SPE joint payoff from any initial state becomes continuous in ε.

Doomed partnerships. Returning to the more general context illustrated in Figure 3, the partnership is "doomed" if both players shirk and then at least one exits at every history in every SPE (in which case the optimal work and exit thresholds $c^{W*} = c^{E*} = 0$). In an unchanging version of the game, the partnership is doomed iff $v > \frac{\delta}{1-\delta}$, since $\sum_{t\geq 1} \delta^t = \frac{\delta}{1-\delta}$ is the continuation payoff generated for each player in a permanently productive relationship. In this paper's changing environment, the partnership is doomed even when the outside option is less valuable than $\frac{\delta}{1-\delta}$. Intuitively, the reason is that partnerships always "break down before they break up". Namely, when the time is reached at which the partnership will end, both players will shirk and earn zero stage-game payoff since they have nothing to gain from exerting effort. If the players had exited earlier, they could have avoided this loss. Thus, in order to be willing to remain in the partnership, players will require that cooperation generate an excess return over their outside option.[11]

Period	2	3	4	5	10	25
% partnership ends	25%	16.7%	12.8%	9.8%	5.0%	2.0%

Table 1: Probability that the partnership ends in period t, when $\log(C_t)$ follows a symmetric random walk, conditional on $c_0 = c^{E*}$ and on survival until time t.

Survivorship bias. Since the partnership ends once the cost of effort first exceeds an exit threshold, the partnerships that have survived several periods will, more likely than not, have received mostly positive shocks that moved the cost of effort away from the exit threshold.

[11]The maximal outside option $\overline{v}$ consistent with any SPE cooperation equals $\frac{\delta - X}{(1-\delta)(1+X)} < \frac{\delta}{1-\delta}$ where $X = \Pr\left(\max_{1\leq s<t} C_s \leq c_0, C_t > c_0 | c_0 = c^{E*}\right) > 0$. The derivation of this formula is omitted to save space.

This positive selection effect tends to make partnerships that have lasted a long time less likely to end in the near future. For example, suppose that $\log(C_t)$ follows a symmetric random walk (as in Example 1) and that $c_0 = c^{E*}$ so that players are just barely willing to stay in the relationship. Table 1 documents the hazard rate of exit over time. For instance, conditional on survival until time $t = 4$, the partnership will end that period approximately 12.8% of the time. The survivorship bias effect is present here, as the probability of exit decreases with age.[12] (The fact that the exit hazard at time t is approximately $\frac{1}{2t}$ follows from the symmetric random walk assumption; see Hughes (1995).)

4 Welfare-maximizing equilibrium

Let $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ denote the maximal joint payoff that can be achieved in *any* subgame-perfect equilibrium (SPE), as evaluated before efforts at time t from payoff-relevant history $(x_t, \mathbf{e}_{t-1})$, given outside options $v = (v_i, v_j)$.[13] In this section, I will demonstrate a SPE that achieves this maximal joint payoff at every history reached on the equilibrium path.

Each player i is only willing to exert effort e_{it} as part of an effort-profile e_t if play (including wage transfers) after time-t efforts will generate a continuation payoff of at least $v_i + c_{it}(e_t; x_t)$. In particular, costly efforts (such that $c_{\Sigma t}(e_t; x_t) > 0$) can only be sustained if joint continuation payoff inside the partnership exceeds players' joint outside option *plus* their joint cost of effort. However, joint inside continuation payoff after efforts e_t is bounded above by $\max\left\{v_\Sigma, \delta E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t\right]\right\}$. Thus, maximal SPE joint payoff $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ before time-t efforts is *at most*

$$\max_{e_t}\left(\pi_{\Sigma t}(e_t; x_t) + \max\left\{v_\Sigma, \delta E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t\right]\right\}\right) \tag{5}$$

$$\text{subject to } c_{\Sigma t}(e; x_t) \leq \max\left\{0, \delta E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t\right] - v_\Sigma\right\} \tag{6}$$

In fact, this upper bound on time-t joint payoff can be realized in SPE, given wage promises that take the form of "retention bonuses", paid after and only if both players decide to stay.

[12] In general, the hazard of exit need not be monotone. For instance, suppose that $\log(C_t)$ is very likely to either fall by slightly less than two or rise by slightly more than one, and that $c_0 = c^{E*}$. Conditional on both staying at time $t = 1$, the partnership is much more likely end at time $t = 3$ then at time $t = 2$.

[13] It is without loss to restrict attention to SPE in which payoffs do not depend on the history of wages.

Lemma 1 (Joint welfare-maximizing SPE play)**.** *Suppose that SPE exist such that, at time* $t+1$ *from each history* $(x_{t+1}, \mathbf{e}_t)$, *players' joint payoff is* $\Pi^{eqm}_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$. *(i) A SPE exists such that, at time t from history* $(x_t, \mathbf{e}_{t-1})$, *players' joint payoff* $\Pi^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ *solves*

$$\max_{e_t} \left(\pi_{\Sigma t}(e_t; x_t) + \max\left\{v_\Sigma, \delta E\left[\Pi^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right]\right\}\right) \tag{7}$$

$$\text{subject to } c_{\Sigma t}(e_t; x_t) \leq \max\left\{0, \delta E\left[\Pi^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right] - v_\Sigma\right\} \tag{8}$$

(ii) In this SPE, wages are paid (only) just after players decide whether to quit.

Proof. Let $\Pi^{eqm}_{it}(x_t, \mathbf{e}_t; v) = \delta E\left[\Pi^{eqm}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right]$ be shorthand for player i's expected time-t continuation payoff, after efforts e_t from history $(x_t, \mathbf{e}_{t-1})$, should it remain active at time $t+1$. Figure 4 illustrates the key idea of Lemma 1. As long as $\Pi^{eqm}_{\Sigma t}(x_t, \mathbf{e}_t; v)$ exceeds the players' joint payoff after exit, v_Σ, *plus* their joint incentive to shirk from efforts e_t, $c_{\Sigma t}(e_t; x_t)$, there exists a retention bonus promise given which both players have sufficient incentives to exert efforts e_t and then stay. Further, this promise is credible since each player promises less than his willingness to pay to avoid cooperation breakdown.

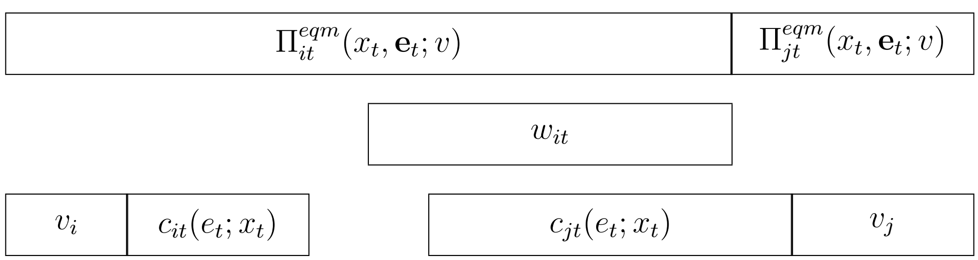

Figure 4: Efforts e_t are incentive-compatible when player i pays wage w_{it} (and $w_{jt} = 0$).

Let $\Delta_{it}(e_t) = \Pi^{eqm}_{it}(x_t, \mathbf{e}_t; v) - v_i - c_{it}(e_t; x_t)$ denote player i's "excess inside continuation payoff", the extra profit that he enjoys inside the partnership after efforts e_t, relative to deviating with zero effort and then quitting the relationship. $\Delta_{it}(e_t)$ is the most that player i can credibly promise to pay player j as a reward for not deviating from the prescribed efforts e_t and then not quitting.[14]

[14] Should efforts e_t be played, player i becomes willing to pay up to $\Delta_{it}(e_t) + c_{it}(e_t; x_t)$ to avoid exit. Then, should both players stay to period $t+1$, player i becomes willing to pay more still to avoid a transition to an optimal punishment continuation SPE in which both players exert zero effort, pay zero wages, and exit for

Without loss, suppose that $\Delta_{it}(e_t) \geq \Delta_{jt}(e_t)$. If $\Delta_{it}(e_t) + \Delta_{jt}(e_t) < 0$, then at least one player must strictly prefer to deviate by exerting zero effort and then quitting, given any credible wage. Otherwise, any retention bonus $w_{it} \in [\max\{0, -\Delta_{jt}(e_t)\}, \Delta_{it}(e_t)]$ from player i to player j can credibly support efforts e_t. Thus, effort-profile e_t can be supported in some SPE iff it satisfies (8). This completes the proof, since then the maximal SPE joint welfare given the specified continuation payoffs is the solution to (7). □

Joint-welfare maximizing SPE. Lemma 1 maps the maximal joint payoff that can be supported at time $t+1$ to the maximal joint payoff that can be supported at time t. Thus, in the spirit of Abreu, Pearce and Stacchetti (1990), one can construct a sequence of upper bounds for every history, $\left\{\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) : k = 1, 2, ...\right\}$, such that $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is non-increasing in k and converges to the maximal SPE joint payoff at history $(x_t, \mathbf{e}_{t-1})$. Also importantly for this paper's purposes, these upper bounds exhibit a monotonicity in the state, i.e. $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is weakly increasing in x_t for all k as well as in the limit.

Theorem 1 (Maximal joint welfare)**.** *The maximal joint welfare* $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ *that can be realized in SPE from history* $(x_t, \mathbf{e}_{t-1})$ *is weakly increasing in* x_t*, for all* $\mathbf{e}_{t-1}$*.*

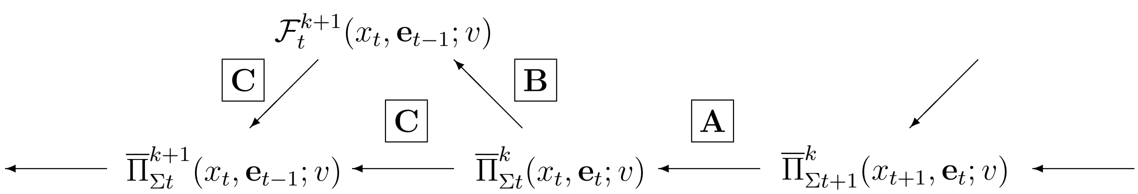

Figure 5: Illustration of the key steps of Theorem 1's proof.

Key steps of the proof of Theorem 1 (Figure 5): Suppose that there exists upper bounds $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ on SPE joint payoff from all histories at all times and that these upper bounds are weakly increasing in the current state x_t. The essence of the proof is to use these upper bounds to derive *weakly lower* upper bounds $\overline{\Pi}^{k+1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ at all histories that remain weakly increasing in x_t. Here in the text, I provide the inductive step to construct this

certain at time $t+1$. Thus, as long as player i has not promised to pay more than $\Delta_{it}(e_t)$, he has sufficient incentive to exert his prescribed effort, then stay, then pay the specified bonus.

sequence of upper bounds on joint payoff and show that monotonicity in x_t is preserved along this sequence. In the Appendix, I prove that this sequence of bounds is weakly decreasing in k and that it converges to joint payoff that can be realized in SPE.

Step A. Given bounds $\overline{\Pi}^k_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$ on joint continuation payoff at time $t+1$, joint continuation payoff after time-t efforts is bounded by

$$\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v) = \max\left\{v_\Sigma, \delta E\left[\overline{\Pi}^k_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right]\right\}. \tag{9}$$

Observe that

$$E\left[\overline{\Pi}^k_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right] = \int_0^\infty \Pr\left(\overline{\Pi}^k_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) \geq z|x_t, \mathbf{e}_t\right) \mathrm{d}z. \tag{10}$$

By presumption, $\overline{\Pi}^k_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$ is weakly increasing in x_{t+1}. Thus, the set $\{x_{t+1} : \overline{\Pi}^k_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) \geq z\}$ is an increasing subset of $\mathcal{X}_{t+1}$ for all z. By Assumption 4, then, each of the probability terms inside the integral in (10) is weakly increasing in x_t. Thus, $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v)$ is weakly increasing in x_t.

Step B. Let $\mathcal{F}^{k+1}_t(x_t, \mathbf{e}_{t-1}; v)$ be the set of time-t efforts that can be supported in SPE given joint continuation payoffs $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v)$ after effort, i.e. those satisfying the IC-constraint (8) given these expected joint continuation payoffs after time-t effort. Since (8) slackens with higher continuation payoffs, the fact that $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v)$ is weakly increasing in x_t implies that $\mathcal{F}^{k+1}_t(x_t, \mathbf{e}_{t-1}; v)$ is weakly increasing in x_t, relative to the set inclusion order.

Step C. By Lemma 1, we may define new upper bounds on time-t SPE joint payoff,

$$\overline{\Pi}^{k+1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) = \max_{e_t \in \mathcal{F}^{k+1}_t(x_t, \mathbf{e}_{t-1}; v)} \left(\pi_{\Sigma t}(e_t; x_t) + \overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v)\right). \tag{11}$$

$\overline{\Pi}^{k+1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is weakly increasing in x_t since both $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_t; v)$ and $\mathcal{F}^{k+1}_t(x_t, \mathbf{e}_{t-1}; v)$ are weakly increasing in x_t, while $\pi_{\Sigma t}(e_t; x_t)$ is weakly increasing in x_t by Assumption 2. $\square$

Finally, Lemma 2 establishes that the maximal SPE joint payoff depends on players' outside options only through their *sum*.

Lemma 2. $v'_\Sigma = v_\Sigma$ *implies* $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v') = \overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$.

Proof. The proof of Lemma 2 is immediate from the algorithmic construction in the proof of Theorem 1. Lemma 2 can also be viewed as a corollary of Lemma 1, once one observes that players' outside options do not appear in the objective (7) or in the constraint (8) except through the sum v_Σ. Intuitively, asymmetries in players' outside options have no impact on what can be achieved in equilibrium, since any such asymmetries can be "counter-balanced" by appropriate wage payments. □

Definition 5 (Maximal SPE joint payoff)**.** Let $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v_\Sigma)$ denote the maximal joint payoff in any SPE from history $(x_t, \mathbf{e}_{t-1})$, as a function of players' joint outside option v_Σ.

5 Re-matching in a partnership economy

This section embeds the partnership game of Section 2 within a "partnership economy". In the context of this economy, I introduce "gender" to capture settings in which players are born into different roles, e.g. buyer and supplier, worker and firm, entrepreneur and investor.

Partnership economy. There is a unit mass of atomless players, half "male" and half "female", with an equal flow of $(1-\delta)$ births and deaths each period. Each player dies with exogenous probability $(1-\delta)$ each period, where death is iid across periods, and seeks to maximize his total undiscounted expected payoff prior to death.

Re-matching. Any player who is newly-born or whose partnership ended at time t (whether due to the death of a partner or due to endogenous exit) is automatically and costlessly re-matched with a new partner at time $t+1$. Further, each such re-matching is a "fresh start" in two senses. First, players know nothing about their current partner's history before their partnership, including his age, number of past partnerships, and so on. Thus, strategies in one partnership cannot be conditioned on what happened in previous partnerships. Second, partnerships are stochastically independent, in the following sense.

Assumption 5 (Fresh start)**.** Let $\{X_t^{ij_1t_1} : t \geq t_1\}$ denote the stochastic state of a potential partnership between players i, j_1 at time t, should they have begun such a partnership at time t_1. For all $i, j_1, j_2, t_2 \leq t_1$, $\{X_{t+t_2-t_1}^{ij_2t_2} : t \geq t_1\}$ is iid as $\{X_t^{ij_1t_1} : t \geq t_1\}$.

Assumption 5 imposes at least two substantive economic restrictions. First, shocks to a partnership are totally idiosyncratic to the players in that partnership.[15] This rules out the possibility of economy-wide shocks (correlated across partnerships active at the same time), which would of course be interesting to study in the context of enriching existing models of the business cycle. Indeed, extending the present analysis to allow for correlated shocks appears to be an important and promising direction for future research. Second, shocks to a partnership have no bearing on future partnerships in which those players might participate. Thus, the "state" here does not capture any payoff-relevant characteristics of the players themselves (such as intelligence, beauty, or skills).

Each individual partnership fits in the model of Section 2, given discount factor δ and exogenous probability of separation $\lambda = 1 - \delta$. (Each player will act as if maximizing discounted payoffs relative to discount factor δ, since he dies with probability $(1 - \delta)$ each period. Each player is exogenously separated iff his current partner dies, which happens with probability $1 - \delta$ conditional on his own survival.) Of course, players' outside options $v = (v_i, v_j)$ are endogenous in the partnership economy, as they depend on how players expect future partnerships to proceed.

Since histories are unobserved, it is without loss to restrict attention to equilibria of the overall partnership economy in which all pairs of players play the same SPE of the partnership game.

Definition 6 (Partnership-economy equilibrium)**.** A *"partnership-economy equilibrium"* is a SPE of the partnership game with the extra property that each player's outside option is the expected present value of his/her option to exit and re-match with a new partner. That is, any SPE that generates payoffs for each player i such that

$$v_i = \delta E\left[\Pi_{i0}^{eqm}(X_0; v)\right]. \tag{12}$$

In particular, a necessary condition of partnership-economy equilibrium is that the players' *joint* outside option can be "self-generated" in SPE (see Lemma 2 and Definition 5):

$$v_\Sigma = \delta E\left[\Pi_{\Sigma 0}^{eqm}(X_0; v_\Sigma)\right]. \tag{13}$$

[15] For all (i_1, j_1) and (i_2, j_2), Assumption 5 requires that $X_t^{(i_1,j_1)}, X_t^{(i_1,j_2)}, X_t^{(i_2,j_2)}$ all be independent.

A social-welfare maximizing partnership-economy equilibrium is one that maximizes players' joint outside option, among all partnership-economy equilibria. In general, such an equilibrium need not maximize joint welfare within each partnership. However, such tension between economy-wide and individual-partnership welfare only arises when the maximal ex ante expected joint payoff that can be supported in SPE is discontinuous, when viewed as a function of the players' joint outside option.

Theorem 2 (Maximal social welfare)**.** *Suppose that* $E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ *is continuous in* v_Σ*. In all social-welfare maximizing partnership-economy equilibria, players' expected joint payoff when first matched is* v^*_Σ/δ*, where* v^*_Σ *is the* unique *solution to (13). Further, players in each partnership play a joint-welfare maximizing SPE given their endogenous outside options.*

Social-welfare maximizing partnership-economy equilibria differ in how the surplus is divided between the players. For example, if a wife anticipates that her *next* husband will pay her a handsome wage, then her current husband may have to pay her a wage to induce her to stay. In this way, even if there exists a social-welfare maximizing partnership-economy equilibrium in which no wages are paid, other such equilibria may exist in which either player receives the lion's share of the surplus. On the other hand, all social-welfare maximizing partnership-economy equilibria specify the same efforts every period[16] and indeed are indistinguishable save by the endogenous division of surplus through wages.

Theorem 2 contains two key observations. First, assuming that players maximize joint payoff within each partnership, there is a *unique* joint outside option v^*_Σ that can be supported in partnership-economy equilibrium. Intuitively, as players' outside options become less valuable, they have more reason to invest in their relationship. Thus, even if falling outside options are bad news in the sense of lowering equilibrium payoffs, this loss is mitigated (and possibly overwhelmed) by the fact that the players' partnership becomes stronger and more productive. Second, in the social-welfare maximizing partnership-economy equilibrium, players within each partnership play the joint-welfare maximizing SPE.

[16]If (5) has multiple solutions at some history, then there will exist different joint-welfare maximizing SPE in which each of these optimal IC efforts is played. Otherwise, the efforts specified in joint-welfare maximizing SPE are unique.

This second key finding is not obvious and, in fact, flies in the face of received wisdom about welfare-maximizing play in repeated games with re-matching. Consider the example of Section 3, in the special case in which c_0 is non-random and $c_t = c_0$ for all t. In the joint-welfare maximizing SPE of this repeated game, players either work and stay forever (if $v \leq \frac{\delta}{1-\delta}$) or shirk and quit immediately (if $v > \frac{\delta}{1-\delta}$). However, as explained in Section 5.2 of Mailath and Samuelson (2006), the social-welfare maximizing partnership-economy equilibrium entails an "incubation period" whereby players shirk and stay for several periods and then work and stay forever. Intuitively, since re-matching is costless and anonymous, each player would gain from being a "serial shirker" if immediate cooperation could be supported in partnership-economy equilibrium.

Theorem 2 shows that this failure of renegotiation-proofness in repeated games with re-matching hinges crucially on the fact that the maximal joint payoff that can be achieved in SPE is discontinuous in the players' endogenous outside option. (In the example, joint payoff $2/(1-\delta)$ can be supported for all $v_\Sigma \leq \frac{\delta}{1-\delta}$ whereas only $2v$ can be supported when $v_\Sigma > \frac{\delta}{1-\delta}$.) Many combinations of assumptions generate the continuity property needed to restore renegotiation-proofness of welfare-maximizing play. For example, suppose that the initial state is augmented with a one-dimensional "partnership type" s_0, defined as follows.

Definition 7 (Effort-incentivizing type)**.** The initial state x_0 includes an "independent effort-incentivizing partnership type" s_0 if (i) $x_t = (s_0, y_t)$ for all $t \geq 0$, (ii) $S_0 \in \mathbf{R}$ is atomless and independent of $(Y_t : t \geq 0)$, and (iii) $\pi_{it}(e'_t; s_0, y_t) - \pi_{it}(e_t; s_0, y_t)$ is strictly increasing in s_0 for all $i, t, y_t, e'_t \succ e_t$.

The partnership type s_0 captures an aspect of "initial fit" between partners in that, all else equal, higher partnership types lead to higher inside payoffs and lower effort costs. (Both $\pi_{it}(e_t; s_0, y_t) - \pi_{it}(0, 0; s_0, y_t) = \pi_{it}(e_t; s_0, y_t)$ and $-c_{it}(e_t; s_0, y_t) = \pi_{it}(e_t; s_0, y_t) - \pi_{it}(0, e_{jt}; s_0, y_t)$ are strictly increasing in s_0, for all t, x_t, e_t.) Since partnership type can have an arbitrarily small impact on payoffs, one may view Claim 2 as establishing the continuity condition of Theorem 2 in a perturbation of the model in which initial fit matters. (The additional assumption of Claim 2, that there are finitely many efforts, is not essential but simplifies and shortens the proof.)

Claim 2. *Suppose that the initial state includes an independent effort-incentivizing partnership type, and that $\mathcal{E}_t$ is finite for all t. Then $E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ is continuous in v_Σ.*

Steady-state distribution over partnership histories. One may view a partnership in the economy as a Markov chain over *histories* $h_t = (x_t; \mathbf{e}_{t-1})$, where any partnership that ends at time t is understood to transition to a brand new partnership.

Suppose that a partnership is currently in history h_t. Let $e_t(h_t)$ be the effort-profile played in the optimal SPE at this history, as characterized in the proof of Theorem 1. Similarly, let $p^{exit}_t(h_t)$ be the probability of endogenous exit, due to at least one player choosing to leave, and let $X_{t+1}(h_t) \sim X_{t+1}|(h_t, e_t(h_t))$ denote next-period's state should the partnership persist to that time. Transition probabilities among histories may be fully described as follows:

- With probability $1 - \delta^2$, the partnership will end due to death, after which a new partnership will be created having random initial history $H_0 = X_0$.
- With probability $\delta^2 p^{exit}_t(x_t, \mathbf{e}_{t-1})$, the partnership will end due to some partner's endogenous departure, after which a new partnership will again be created.
- With probability $(1-\delta^2)(1-p^{exit}_t(x_t, \mathbf{e}_{t-1}))$, the partnership will continue to time $t+1$, with an augmented random history $H_{t+1} = (h_t; e_t(h_t); X_{t+1}(h_t))$.

Note that, through the process of death and re-birth, all histories that are reached on the equilibrium path communicate and are positively recurrent. Thus, this Markov chain is ergodic and there exists a unique steady-state distribution over histories. (The proof of Lemma 3 is omitted to save space. See Sections 4.3 and 4.6.2 of Ross (1996), especially Theorem 4.3.3.)

Lemma 3 (Steady-state distribution)**.** *For every partnership-economy equilibrium, there exists a unique steady-state distribution over partnership histories.*

In the remainder of this section, I discuss some qualitative features of a "typical" player's life experience, assuming play of a welfare-maximizing partnership-economy equilibria.

Dating. At time $t = 0$, players will immediately exit any relationship in which the realized initial state is in a decreasing subset of $\mathcal{X}_0$. Consequently, any player who is seeking a new partner will typically experience several partnerships that each last exactly one period – and in which both players exert zero effort because they anticipate no future interaction – before finding a partner who they do not immediately leave.

Honeymoon. in any partnership that continues to a second period, players obviously expect higher continuation payoffs than during their unsuccessful dating phase. In fact, such "newly-joined" partners will also enjoy higher stage-game payoffs than when they were unsuccessfully dating, for two reasons. First, the initial state in a "successful date" will be higher than in an unsuccessful one, allowing players to generate higher stage-game payoffs from any time-0 efforts (Assumption 2). Second, since the players view their future relationship as generating higher continuation payoffs than their outside options, they can also support non-trivial effort at time $t = 0$.

There is no guarantee that a partnership in its "honeymoon" will be very profitable or very stable. For instance, it could be that the initial state lies very close to the threshold below which the partnership would not have formed, and that there is a high likelihood of break-up in the near future. On the other hand, depending of course on the distribution of the initial state, a large fraction of new partnerships may have initial states far enough above this threshold so that exit is very unlikely for several periods. If so, one would observe a "dating and honeymoon effect" in which partnerships are very likely to end in their first period, very unlikely to end in their second period, and then somewhat more likely to end over the next several periods. The honeymoon effect has also been articulated by Fichman and Levinthal (1991) in a non-strategic context.[17]

Hard times. The state of a partnership may rise and fall many times, in ways that affect the extent of cooperation that can be supported in the welfare-maximizing equilibrium. This volatility of players' willingness to cooperate creates payoff volatility that in turn creates an

[17]Fichman and Levinthal (1991) consider an organization's decision to form and disband, when profits follow an exogenous random walk. The analysis here differs in several ways, the most important being that profits are endogenous.

endogenous option value to remaining in the relationship. Consequently, players tend to remain in relationships even when stage-game payoffs are low, in hopes that their partner's behavior will improve.

This feature of equilibrium behavior appears consistent with an interesting fact about marital separation in the United States, if one can interpret reports of "happiness" as reflecting stage-game payoffs. The National Survey of Families and Households (NSFH) of 1987-1988 asked about two thousand individuals who had experienced marital separation relatively recently to evaluate their experience.[18] When comparing their overall happiness "now, compared to the year before you separated", 57.8% described their current happiness as "much better" while only 2.9% described it as "much worse." One possible explanation of this survey result is that spouses' expectations regarding the future performance of a marriage (relative to their outside options) changes over time, so that there is an option value of staying married.[19] This option value could arise from exogenous variation in the fundamentals of the marriage ("perhaps my wife will get a raise") and/or from endogenous variation in spousal behavior ("perhaps my husband will stop cheating on me"). This paper's analysis shows how strategic behavior can *amplify* the option-value effects due to exogenous variation ("perhaps I will get a raise, after which my husband will stop cheating on me").

Good times and golden years. Players stay in the partnership during hard times in the hope that they will enjoy positive shocks that will enable them to enjoy higher profits and greater stability in the future. Indeed, depending on the details of the stochastic process $(X_t : t \geq 0)$, there may be an increasing subset of the state-space from which the partnership is certain never to end, save by exogenous death. Such "golden years" can arise for two sorts of reasons. First, there may be an absorbing portion of the state-space, that is everywhere high enough to support continuation of the partnership. Second, equilibrium efforts in high enough states may be sufficiently high and feedback from profitable efforts may be positive

[18] All respondents had experienced a separation since January 1, 1977. See www.ssc.wisc.edu/nsfh, accessed June 4, 2009.

[19] Other plausible explanations have nothing to do with uncertainty about how the marriage will perform, such as selection bias (if happier individuals are more likely to be surveyed) or choice-supportive bias (if respondents tend to remember the past in ways that help justify their decisions).

enough to overwhelm any exogenous shocks that might cause the relationship to deteriorate.

6 Comparative statics

The model imposes essentially no restriction on how efforts can control the stochastic process driving inside payoffs. Consequently, there is little that one can say in general about how efforts in the welfare-maximizing SPE vary with the state, nor on how the history of efforts impacts equilibrium variables such as players' payoffs, efforts, and exit.[20] Indeed, although Theorem 1 established that players' joint payoff in the joint-welfare maximizing SPE is weakly increasing in the state x_t, neither realized joint stage-game payoff nor joint continuation payoff need be weakly increasing in x_t. Consequently, partners may exert lower efforts and even be more likely to exit in higher states. However, additional comparative statics are available in a notable special case, when players' efforts have *no impact* on the future distribution of payoffs.

Definition 8 (Exogenous stochastic process)**.** $(X_t : t \geq 0)$ is an *exogenous stochastic process* if the distribution of X_t depends only on (t, x_{t-1}).[21]

Given exogeneity, players' effort-decisions at time t have no impact on the set of SPE continuation payoffs. Thus, in any joint-welfare maximizing SPE, players will choose whatever efforts maximize joint stage-game payoff, among those satisfying the relevant incentive-compatibility constraint.

Theorem 3 (Comparative statics with an exogenous state)**.** *Suppose that* $(X_t : t \geq 0)$ *is an exogenous stochastic process. In the joint-welfare maximizing SPE, at every history reached on the equilibrium path: (i) players' joint stage-game payoff and joint continuation payoff is weakly increasing in* x_t*; and (ii) partnership stopping time conditional on* x_t *is weakly increasing in* x_t*, in the sense of first-order stochastic dominance.*

[20] In various special cases of the model, not described here because of space restrictions, one can establish additional comparative statics vis-a-vis equilibrium effort and its impact on other equilibrium variables.

[21] The current state may depend on the full history of past states. For example, if $x_t = (x_{t-1}, y_t)$ for all $t > 0$, then the distribution of X_t can depend on all of the "new information" $(x_0, y_1, ..., y_{t-1})$ learned during the course of the partnership.

Suppose that $(X_t : t \geq 0)$ is an exogenous stochastic process and that $x_t \in \mathbb{R}$ for all t. An immediate and potentially testable corollary of Theorem 3 is that partnership stopping time conditional on players' realized time-t joint stage-game payoff $\pi_{\Sigma t}$ is weakly increasing in $\pi_{\Sigma t}$, in the sense of first-order stochastic dominance.

7 Concluding Remarks

A broad empirical literature from Topel and Ward (1992) on employment, Levinthal (1991) on firm survival, and Stevenson and Wolfers (2007) on marriage have established certain stylized facts about relationship dynamics. For example, one common thread is that partnerships tend to exhibit a "survivorship bias" in that older partnerships are often more profitable and less likely to end in the near future.

Theory has offered several sorts of explanations of the survivorship bias. (i) *Incomplete information:* Players learn about each other through actions and partnership survival is associated with a positive selection. See e.g. Ghosh and Ray (1996) in which discount rates are private and Watson (1999) with private stage-game payoffs. (ii) *Exogenous shocks:* Payoffs are subject to exogenous shocks, with partnership survival again associated with a positive selection. See e.g. Jovanovic (1979a) in which a worker learns about the productivity of his match with a firm. (iii) *Endogenous investment:* Relationships that survive are more likely to be those in which players have and will continue to invest. See e.g. Jovanovic (1979b) in which a worker can increase his current match profits at a cost. A fourth important type of theory has other implications for partnership dynamics. (iv) *Imperfect monitoring:* Play must occasionally transition from cooperative to non-cooperative phases to overcome an incentive problem created by imperfect monitoring. See e.g. Green and Porter (1984).

This paper abstracts from issues of incomplete information and imperfect monitoring, and instead adds to the literature on exogenous shocks and endogenous investment. I develop a flexible and tractable model of complete-information partnerships with stochastic payoffs and endogenous exit, in which welfare-maximizing equilibria are relatively simple and can generate a variety of comparative statics. This model is then used to derive the steady-state properties of an economy of partnerships with anonymous re-matching.

The theory developed here has different implications for partnership dynamics than existing theories. For instance, consider the patterns of effort that arise in the example of Section 3, in the special case without exit (when players' outside option is zero), compared with those in Ghosh and Ray (1996) or in Green and Porter (1984). Here, partners sometimes shirk and sometimes work. Further, stochastic transitions between work and shirk regimes have the following features (among others): (a) "regime persistence", the likelihood that partners will switch from working to shirking or vice versa tends to decrease the longer that they have continued the same behavior; (b) "back-tracking", while partnerships become more cooperative over time on average, individual partnerships may become less cooperative for a while, then return to a more cooperative footing for a while, and so on. Ghosh and Ray (1996)'s incomplete information model exhibits regime persistence but not back-tracking. In their model, all partnerships that have lasted longer are less likely to break down than all brand-new partnerships. On the other hand, Green and Porter (1984)'s imperfect information model exhibits backtracking but not regime persistence. Indeed, given a fixed monitoring technology, a cooperative partnership will enter a punishment phase with constant rather than declining probability. (Also, transitions back to the cooperative regime violate regime persistence if the punishment regime is of fixed length.)

I conclude with a brief discussion of a few interesting potential directions for future work.

Macroeconomic volatility. One promising direction for future work would be to consider the macroeconomic implications of this model of the economy, allowing for the more general possibility that performance is correlated across active partnerships and that re-matching is costly. For example, suppose that different partnerships that are active at the same time are subject to common shocks as well as idiosyncratic private shocks, but that the cost of re-matching does not change over time. One conjecture in this context is that equilibrium partnership dynamics will have a dampening effect on macroeconomic shocks. After a string of positive common shocks, partnerships will generally be very profitable, potentially making the cost of re-matching negligible relative to the productivity differences across partnerships. In this case, partnerships will tend to be less stable, dampening the benefits of positive common shocks. On the other hand, after a string of negative shocks, the cost of re-matching may loom large enough that players' outside option may simply be to go out of business. In this case, current partners have more incentive to stay and work together, dampening the negative effect of negative common shocks.

Changing individuals. In this paper, each player's next partnership is stochastically identical to his current one. In other words, all shocks are to partnerships, not to the individuals in those partnerships. Of course, individuals may also change in ways that will persist in a new match. Enriching the model to allow for such personal characteristics is important and could have profound implications for the steady-state distribution of the partnership economy. For one thing, the set of players seeking to re-match will be adversely selected. This could increase active partners' desire to avoid the re-matching market, creating a still deeper adverse selection in this market.

Endogenous learning. The model here can capture a wide array of "learning" settings in the spirit of Jovanovic (1979a), including ones in which players make investments to increase the precision of a public signal about an unobserved payoff-relevant parameter. (Such investment could be one aspect of players' multi-dimensional effort.) When investing in a more precise signal of the underlying state, players create short-term volatility in their beliefs about the state. Such volatility can increase the value of the players' option to exit, but could also be harmful if it disrupts an otherwise productive partnership. This suggests a speculation, that players in a stable relationship may actively seek to avoid uncovering new information, while players in a rocky relationship may seek to uncover as much new information as possible.

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Appendix

Proof of Theorem 1.

Proof. Let $\overline{\Pi}_t^{eqm}(x_t, \mathbf{e}_{t-1}; v) \in \mathbf{R}^2$ be the players' payoff profile in a SPE that maximizes *joint* welfare among all SPE from history $(x_t, \mathbf{e}_{t-1})$, and let $\overline{\Pi}_{\Sigma t}^{eqm}(x_t, \mathbf{e}_{t-1}; v) = \overline{\Pi}_{\Sigma t}^{eqm}(x_t, \mathbf{e}_{t-1}; v)$.

Outline of proof. I construct a monotonically decreasing sequence of bounds on SPE joint welfare from each history, $(\overline{\Pi}_{\Sigma t}^{k}(x_t, \mathbf{e}_{t-1}; v) : k \geq 0)$, that converges pointwise to $\overline{\Pi}_{\Sigma t}^{eqm}(x_t, \mathbf{e}_{t-1}; v)$. Further, $\overline{\Pi}_{\Sigma t}^{k}(x_t, \mathbf{e}_{t-1}; v)$ is non-decreasing in x_t for each k, as well as in the limit $\overline{\Pi}_{\Sigma t}^{eqm}(x_t, \mathbf{e}_{t-1}; v)$.

Part I: Decreasing sequence of bounding payoff-profile sets. By Assumption 1, there exists a uniform upper bound M on players' joint payoff at any history. Define $\overline{\Pi}_{\Sigma t}^{0}(x_t, \mathbf{e}_{t-1}; v) = M$

at all histories. Clearly, $\overline{\Pi}^0_{\Sigma t} \geq \overline{\Pi}^{eqm}_{\Sigma t}$. Next, for all $k \geq 1$, define $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ recursively as follows (using shorthand $\mathbf{e}_t = (\mathbf{e}_{t-1}, e_t)$):

$$\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) = \max_{e_t} \left(\pi_{\Sigma t}(e_t; x_t) + \max \left\{ v_\Sigma, \delta E \left[\overline{\Pi}^{k-1}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] \right\} \right) \tag{14}$$

$$\text{subject to } c_{\Sigma t}(e_t; x_t) \leq \max \left\{ 0, \delta E \left[\overline{\Pi}^{k-1}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] - v_\Sigma \right\} \tag{15}$$

Assuming that $\overline{\Pi}^{k-1}_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$ are upper bounds on joint payoff at time $t+1$, then (15) is a necessary condition for efforts e_t to be supported in any SPE from history $(x_{t+1}, \mathbf{e}_t)$. *Proof:* Players at time t expect joint inside continuation payoff of at most $\delta E \left[\overline{\Pi}^{k-1}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right]$ should they choose effort-profile e_t. If players' joint outside option v_Σ exceeds this bound, then at least one player strictly prefers to quit and neither player can be incentivized to exert any costly effort. Otherwise, players' joint cost of effort $c_{\Sigma t}(e_t; x_t)$ must be less than or equal to the amount by which their joint inside continuation payoff exceeds their joint outside option. (Otherwise, at least one player would strictly prefer to deviate by exerting zero effort and then quitting.)

Indeed, (15) is also a sufficient condition to support time-t efforts e_t in SPE given continuation payoffs $\overline{\Pi}^{k-1}_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$. *Proof:* Players are willing to exert efforts e_t and then receive net wage $z_{it} = w_{jt} - w_{it}$, rather than exerting zero effort, receiving zero net wage, and quitting, as long as $c_{it}(e_t; x_t) \leq E \left[\overline{\Pi}^{k-1}_{it+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] - v_\Sigma - z_{it}$ for all i. In particular, efforts e_t are supported by a credible promise of net wage

$$z_{it} = \frac{\left(\delta E \left[\overline{\Pi}^{k-1}_{jt+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] - v_j - c_{jt}(e_t; x_t) \right) - \left(\delta E \left[\overline{\Pi}^{k-1}_{it+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] - v_i - c_{it}(e_t; x_t) \right)}{2}$$

Given these wages (received only after both players commit to remain in the relationship), each player i prefers not to quit since, given (15),

$$\delta E \left[\overline{\Pi}^{k-1}_{it+1}(X_{t+1}, \mathbf{e}_t; v) | x_t, \mathbf{e}_t \right] + z_{it} \geq v_i + c_{it}(e_t; x_t) \geq v_i$$

Since $\overline{\Pi}^0_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ are uniform upper bounds on joint payoff, $\overline{\Pi}^0_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) \geq \overline{\Pi}^1_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$. By induction, $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is non-increasing in k. (The value of the maximization (14) is non-decreasing in continuation payoffs. Thus, $\overline{\Pi}^k_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v) \leq \overline{\Pi}^{k-1}_{\Sigma t+1}(x_{t+1}, \mathbf{e}_t; v)$ for all $(x_{t+1}, \mathbf{e}_t)$ implies $\overline{\Pi}^{k+1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) \leq \overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$.) Further, by induction, $\overline{\Pi}^k_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) \geq$

$\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ for all k. (Higher-than-equilibrium payoffs can be supported given higher-than-equilibrium continuation payoffs. Thus, the fact that $\overline{\Pi}^{k-1}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) \geq \overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ implies $\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v) \geq \overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$.)

Part II: These upper bounds on joint welfare are non-decreasing in x_t.

Base step: $k = 0$. $\overline{\Pi}^{0}_{t}(x_t, \mathbf{e}_{t-1}; v)$ is constant and hence trivially non-decreasing in x_t.

Induction step: $k \geq 1$. Suppose that $\overline{\Pi}^{k-1}_{t}(x_t, \mathbf{e}_{t-1}; v)$ is non-decreasing in x_t for all t. Observe that, for any $x^H_t \succeq x^L_t$,

$$\begin{aligned} E[\overline{\Pi}^{k-1}_{\Sigma t+1}(X_{t+1}; \mathbf{e}_t; v)|x^H_t, \mathbf{e}_t]) &= \int_0^\infty \Pr(\overline{\Pi}^{k-1}_{\Sigma t+1}(X_{t+1}; \mathbf{e}_t; v) > z|x^H_t, \mathbf{e}_t; v)\mathrm{d}z \\ &\geq \int_0^\infty \Pr(\overline{\Pi}^{k-1}_{\Sigma t+1}(X_{t+1}; \mathbf{e}_t; v) > z|x^L_t, \mathbf{e}_t; v)\mathrm{d}z \qquad (16) \\ &= E[\overline{\Pi}^{k-1}_{\Sigma t+1}(X_{t+1}; \mathbf{e}_t; v)|x^L_t, \mathbf{e}_t]) \end{aligned}$$

By the induction hypothesis, $\{x_{t+1} \in \mathcal{X}_{t+1} : \overline{\Pi}^{k-1}_{t+1}(X_{t+1}; \mathbf{e}_t; v) > z\}$ is an increasing subset of $\mathcal{X}_{t+1}$ for all z. Inequality (16) now follows from Assumption 4. Thus, for any given effort-history $\mathbf{e}_t$, $\max\left\{v_\Sigma, \delta E\left[\overline{\Pi}^{k-1}_{\Sigma t+1}(X_{t+1}, \mathbf{e}_t; v)|x_t, \mathbf{e}_t\right]\right\}$ is non-decreasing in x_t, so that higher x_t slackens the IC-constraint (15) while increasing the second term of (14). Finally, the first term of (14) is non-decreasing in x_t by Assumption 2. All together, we conclude that the value of the maximization (14) is non-decreasing in x_t. This completes the desired induction.

Let $\overline{\Pi}^{\infty}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ denote the pointwise limit of $\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ as $k \to \infty$. Since $\overline{\Pi}^{k}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ is non-decreasing in x_t for all k, $\overline{\Pi}^{\infty}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ inherits this monotonicity as well (given the continuity of stage-game payoffs imposed by Assumption 1).

Part III: Limit of upper bounds can be achieved in SPE. It suffices now to show that $\overline{\Pi}^{\infty}_{t}(x_t, \mathbf{e}_{t-1}; v) = \overline{\Pi}^{eqm}_{t}(x_t, \mathbf{e}_{t-1}; v)$. As shown earlier, $\overline{\Pi}^{\infty}_{t}(x_t, \mathbf{e}_{t-1}; v) \geq \overline{\Pi}^{eqm}_{t}(x_t, \mathbf{e}_{t-1}; v)$. Let $e_t(x_t, \mathbf{e}_{t-1})$ denote a limit of any sequence of solutions to (14) subject to (15), as $k \to \infty$. By construction (and assumed continuity of stage-game payoffs), efforts $e_t(x_t, \mathbf{e}_{t-1})$ are incentive-compatible if players expect continuation play in later periods that generates time-$(t+1)$ payoffs of $\overline{\Pi}^{\infty}_{t+1}(x_{t+1}, \mathbf{e}_t; v)$ for each player. Again by construction, these efforts generate continuation payoffs $\overline{\Pi}^{\infty}_{t+1}(x_{t+1}, \mathbf{e}_t; v)$; thus, these strategies constitute a welfare-maximizing SPE. Thus, $\overline{\Pi}^{\infty}_{t}(x_t, \mathbf{e}_{t-1}; v) \leq \overline{\Pi}^{eqm}_{t}(x_t, \mathbf{e}_{t-1}; v)$. This completes the proof. □

Proof of Theorem 2

Proof. First, the bulk of the proof establishes that (13) has a unique solution. Second, this solution corresponds to a social-welfare maximizing partnership-economy equilibrium.

Part I: $v_\Sigma - E[\delta\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ *is strictly increasing in* v_Σ. I will show the stronger result that $v_\Sigma - E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ is non-decreasing in v_Σ.

In a slight variation on the notation used in the text, let $\overline{\Pi}^{eqm}_{\Sigma t}(h_t; v^h_\Sigma)$ denote the maximal SPE joint payoff from history $h_t = (x_t, \mathbf{e}_{t-1})$ given joint outside option v^h_Σ. Consider now a lower joint outside option $v^l_\Sigma \in [0, v^h_\Sigma)$ and let $\Pi_{\Sigma t}(h_t; v^l_\Sigma)$ denote the joint payoff that would result should players with joint outside option v^l_Σ mimic welfare-maximizing play as if it were v^h_Σ. Note that the stage-game payoff process and the partnership stopping time T are identically distributed when players follow the same strategies. Thus, the only difference in payoffs arises from the fact that players only get v^l_Σ when the partnership ends instead of v^h_Σ. In particular, for all histories h_t,

$$\overline{\Pi}^{eqm}_{\Sigma t}(h_t; v^h_\Sigma) - \Pi_{\Sigma t}(h_t; v^l_\Sigma) = (v^h_\Sigma - v^l_\Sigma) \sum_{t' \geq t} \delta^{t'-t} \Pr(T = t' | h_t) \leq v^h_\Sigma - v^l_\Sigma \tag{17}$$

Let $e_t(v^h_\Sigma)$ denote the efforts played in the optimal SPE given joint outside option v^h_Σ. Observe that these efforts remain incentive-compatible given lower joint outside option v^l_Σ:

$$\begin{aligned} E\left[\Pi_{\Sigma t+1}(H_{t+1}; v^h_\Sigma) | h_t, e_t(v^h_\Sigma)\right] &\geq E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(H_{t+1}; v^h_\Sigma) | h_t, e_t(v^h_\Sigma)\right] - \left(v^h_\Sigma - v^l_\Sigma\right) && (18) \\ &\geq \frac{v^h_\Sigma + c_{\Sigma t}(e_t(v^h_\Sigma); x_t)}{\delta} - \left(v^h_\Sigma - v^l_\Sigma\right) && (19) \\ &> c_{\Sigma t}(e_t(v^h_\Sigma); x_t) + v^l_\Sigma \end{aligned}$$

(18) follows from (17). (19) follows from the incentive-compatibility constraint (8) as applied to the optimal equilibrium given v^h_Σ, for any efforts having non-zero cost. (Mimicking zero-cost efforts is trivial in SPE regardless of outside options.) By similar logic, staying is incentive-compatible given these mimicking strategies whenever players stay in the optimal equilibrium given joint outside option v^h_Σ. (Details omitted to save space.) Thus, these mimicking strategies constitute a SPE given v^l_Σ. In particular, $E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v^l_\Sigma)] \geq E[\Pi_0(X_0; v^l_\Sigma)]$. Thus, $E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v^h_\Sigma)] - E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v^l_\Sigma)] \leq v^h_\Sigma - v^l_\Sigma$ as desired.

Part II: $v_\Sigma = E[\delta\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ *has a unique solution* v^*_Σ*.* $E[\delta\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; 0)] \geq 0$ since each player can guarantee at least zero payoff by exerting zero effort every period (Assumption 1). On the other hand, since joint stage-game payoff is uniformly bounded (again by Assumption 1), there exists some outside option $\overline{v}$ such that $E[\delta\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; \overline{v})] < \overline{v}$ regardless of players' efforts. Existence and uniqueness of a solution to $v_\Sigma - E[\delta\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)] = 0$ now follows immediately from the fact that this difference is continuous (by assumption) and strictly increasing in v_Σ (by Part I).

Part III: v^*_Σ *is the maximal joint outside option in any partnership-economy equilibrium.* Given any joint outside option $\overline{v}_\Sigma > v^*_\Sigma$, players' (discounted) expected equilibrium joint payoff should they exit and re-match is at most $E[\delta\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; \overline{v}_\Sigma)] < \overline{v}_\Sigma$. Thus, joint outside option $\overline{v}_\Sigma$ cannot be supported in any partnership-economy equilibrium. However, I have already shown that v^*_Σ can be supported in equilibrium. □

Proof of Claim 2

Proof. Part I: Preliminaries. Since $x_0 = (s_o, y_0)$, continuity of $E[\overline{\Pi}^{eqm}_{\Sigma 0}(X_0; v_\Sigma)]$ in v_Σ follows from continuity of $E[\overline{\Pi}^{eqm}_{\Sigma 0}(S_0, y_0; v_\Sigma)]$ for all y_0. Thus, without loss I will focus on the special case in which the initial state is simply the partnership type, $X_0 = S_0$.

Maximal SPE ex ante expected joint payoff may be expressed as

$$E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(S_0, ; v_\Sigma)\right] = \sum_{t \geq 0} \delta^t \Bigg(\Pr(T \geq t) E\left[\pi_{\Sigma t}(e_t(H_t; v_\Sigma); h_t) | T \geq t\right] + v_\Sigma \Pr(T = t) \Bigg)$$

where T is the random stopping time of the partnership in the optimal SPE (which depends on v_Σ), and $e_t(h_t; v_\Sigma)$ is the prescribed effort-profile in the optimal SPE in history $h_t = (x_t, \mathbf{e}_{t-1})$. Recall that $e_t(h_t; v_\Sigma)$ maximizes joint payoff subject to the IC-constraint that joint continuation payoff is greater than or equal to joint outside option plus joint cost of effort:[22]

$$c_{\Sigma t}(e_t(h_t); h_t) \leq \delta E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(h_t, e_t(h_t; v_\Sigma), X_{t+1}; v_\Sigma)\right] - v_\Sigma \tag{20}$$

[22]To simplify the presentation, I focus on the case in which there is a unique such maximizer at almost all histories reached on the equilibrium path. More generally, the proof extends almost unchanged, when one recognizes that a discontinuity of $E\left[\overline{\Pi}^{eqm}_{\Sigma 0}(S_0, ; v_\Sigma)\right]$ in v_Σ requires that the IC-constraint be binding on *all* such maximizers at a set of histories reached with positive probability.

An increase in joint outside option from v_Σ to $v_\Sigma + \varepsilon$ has two effects on the maximal SPE joint payoff. First, the direct effect is that players enjoy higher joint payoff when quitting and quit whenever they were previously almost indifferent to doing so. This direct effect increases joint payoff by at most ε. Second, since $\delta E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(h_t, e_t(h_t; v_\Sigma), X_{t+1}; v_\Sigma)\right] - v_\Sigma$ is strictly decreasing in v_Σ (see the proof of Theorem 2), an indirect effect is that players cannot support as many effort-profiles at some histories. This decreases payoffs at those histories, inducing more exit and less effort at previous histories, and so on in a backward cascade that decreases joint payoff and may do so discontinuously.

Part II: (20) is binding with zero probability. Fix any joint outside optin v_Σ, effort-profile e_t, effort-profile history $\mathbf{e}_{t-1}$, and sequence of states $\mathbf{x}_{1\to t} = (x_1, ..., x_t)$ realized after time 0. By assumption, $c_{it}(e_t; s_0, \mathbf{x}_{1\to t})$ is strictly decreasing in s_0 for each player i while, by the proof of Theorem 1, $E\left[\overline{\Pi}^{eqm}_{\Sigma t+1}(s_0, \mathbf{x}_{1\to t}, \mathbf{e}_{t-1}, e_t, X_{t+1}; v_\Sigma)\right]$ is weakly increasing in s_0. Thus, if the IC-constraint (20) binds for some efforts e_t at history $(s_0, \mathbf{x}_{1\to t}, \mathbf{e}_{t-1})$, then for all $s_0^l < s_0 < s_0^h$ it fails at history $(s_0^l, \mathbf{x}_{1\to t}, \mathbf{e}_{t-1})$ and is strictly satisfied at history $(s_0^h, \mathbf{x}_{1\to t}, \mathbf{e}_{t-1})$. In particular, let $e_t(h_t)$ denote the effort-profile prescribed in the joint-welfare maximizing SPE from history h_t. Then (20) binds on $e_t(s_0, \mathbf{x}_{1\to t}, \mathbf{e}_{t-1})$ for a zero-measure set of types $s_0 \in \mathbf{R}$. We conclude that, with probability one in the joint-welfare maximizing SPE, the IC-constraint will not be binding on *any* effort-profile prescribed on the equilibrium path.

Part III: Non-binding (20) implies that maximal SPE joint payoff is continuous in v_Σ. I will prove right-continuity here, that $\lim_{\varepsilon\to 0} \overline{\Pi}^{eqm}_{\Sigma t}(h_t; v_\Sigma + \varepsilon) = \overline{\Pi}^{eqm}_{\Sigma t}(h_t; v_\Sigma)$ for all v_Σ and all histories h_t reached with probability one on the equilibrium path. The proof of left-continuity is similar, and omitted to save space.

For this final step, I employ a variation on the algorithm used in the proof of Theorem 1. Fix $\hat{v}_\Sigma$. For all histories h_t and $\varepsilon \geq 0$, define

$$\overline{\Pi}^{1}_{\Sigma t}(h_t; \hat{v}_\Sigma + \varepsilon) = \overline{\Pi}^{eqm}_{\Sigma t}(h_t; \hat{v}_\Sigma) + \varepsilon$$

Since the positive "direct effect" of higher joint outside option discussed earlier is at most ε and the "indirect effect" is always negative, $\overline{\Pi}^{1}_{\Sigma t}(h_t; \hat{v}_\Sigma + \varepsilon) > \overline{\Pi}^{eqm}_{\Sigma t}(h_t; \hat{v}_\Sigma + \varepsilon)$. Also, clearly, $\overline{\Pi}^{1}_{\Sigma t}(h_t; v_\Sigma)$ is continuous in v_Σ for all t and all histories h_t.

Next, as in Steps A-C of the algorithm illustrated in Figure 5, define

$$\overline{\Pi}^1_{\Sigma t}(h_t, e_t; v_\Sigma) = \max\left\{v_\Sigma, \delta E\left[\overline{\Pi}^1_{\Sigma t+1}(h_t, e_t, X_{t+1}; v_\Sigma)|h_t, e_t\right]\right\}$$
$$\mathcal{F}^2_t(h_t; v_\Sigma) = \left\{e_t : v_\Sigma + c_{\Sigma t}(e_t; x_t) \le \overline{\Pi}^1_{\Sigma t}(h_t, e_t; v_\Sigma)\right\}$$
$$\overline{\Pi}^2_{\Sigma t}(h_t; v_\Sigma) = \max_{e_t \in \mathcal{F}^2_t(h_t; v_\Sigma)} \left(\pi_{\Sigma t}(e_t; x_t) + \overline{\Pi}^1_{\Sigma t}(h_t, e_t; v_\Sigma)\right)$$

As already shown, $\overline{\Pi}^1_{\Sigma t+1}(h_t, e_t, x_{t+1}; v_\Sigma)$ is continuous in v_Σ for all histories $h_{t+1} = (h_t, e_t, x_{t+1})$. Thus, $\overline{\Pi}^1_{\Sigma t}(h_t, e_t; v_\Sigma)$ is continuous in v_Σ for all t, h_t as well. Next, since the set of effort-profiles is finite, Part II above implies that the IC-constraint is not (exactly) binding for *any* effort-profile at a probability-one set of histories reached on the equilibrium path. At all such histories, $\mathcal{F}^2_t(h_t; v_\Sigma)$ is unchanging in a neighborhood of $\hat{v}_\Sigma$. We conclude that, at a probability-one set of equilibrium histories, $\overline{\Pi}^2_{\Sigma t}(h_t; v_\Sigma)$ is continuous in v_Σ at $\hat{v}_\Sigma$.

Repeating this argument for all $k \ge 1$, we conclude that $\overline{\Pi}^k_{\Sigma t}(h_t; v_\Sigma)$ is continuous in v_Σ at $\hat{v}_\Sigma$ at a probability-one set of equilibrium histories. Such continuity carries over to the limit as well, so that maximal equilibrium joint payoff $\overline{\Pi}^{eqm}_{\Sigma t}(h_t; v_\Sigma)$ is continuous in v_Σ at a probability-one set of histories. In particular, $E[\overline{\Pi}^{eqm}_{\Sigma t}(S_0; v_\Sigma)$ is continuous in v_Σ at $\hat{v}_\Sigma$. □

Proof of Theorem 3

By Theorem 1, joint payoff $\overline{\Pi}^{eqm}_{\Sigma t}(x_t, \mathbf{e}_{t-1}; v)$ in the joint-welfare maximizing SPE is weakly increasing in x_t for all $(\mathbf{e}_{t-1}, v)$. Given an exogenous stochastic process, further, such payoffs do not depend on the history of efforts. Since the outside option v is held fixed, I will henceforth use the simpler notation $\overline{\Pi}^{eqm}_{\Sigma t}(x_t)$ here.

Proof of (i). Recall that $\overline{\Pi}^{eqm}_{\Sigma t}(x_t) = \max_{e_t}\left(\pi_{\Sigma t}(e_t; x_t) + E\left[\delta\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1})|x_t\right]\right)$ subject to the IC-constraint $E\left[\delta\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1})|x_t\right] \ge c_{\Sigma t}(e_t; x_t) + 2v$. Joint continuation payoff

$$E\left[\delta\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1})|x_t\right] = \int_0^\infty \Pr\left(\delta\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1}) > z|x_t\right) \mathrm{d}z \qquad (21)$$

is weakly increasing in x_t: $\{x_{t+1} : \delta\overline{\Pi}^{eqm}_{\Sigma t+1}(X_{t+1}) > z\}$ is an increasing subset of $\mathcal{X}_{t+1}$ so that, by Assumption 4, each of the probability terms in (21) is weakly increasing in x_t. Finally, since efforts do not control future payoffs, time-t efforts in the optimal SPE will be chosen to maximize joint stage-game payoff subject to the IC-constraint. Since joint continuation payoff is weakly increasing in x_t, so is the set of effort-profiles e_t satisfying the IC-constraint. Consequently, realized joint stage-game payoff is weakly increasing in x_t.

Proof of (ii). In the optimal SPE, the partnership ends in the first period in which joint continuation payoff is less than $2v$ (and joint continuation payoff depends only on the current state). Let $p_t^k(x_t)$ denote the probability that the partnership will end at time $t+k$, given that it is active at time t in state x_t. I need to show that, for each $k \geq 1$, $p_t^k(x_t)$ is weakly increasing in x_t. The proof is by induction.

Base step. By Theorem 3(i), $p_{t'}^1(x_{t'})$ is weakly increasing in $x_{t'}$, including for $t' = t+k-1$.

Induction step. As the induction hypothesis, suppose that the following is true for some $t'+1 \in [t+1, t+k-1]$:

- $p_{t'+1}^m(x_{t'+1})$ is weakly increasing in $x_{t'+1}$ for all $m = 1, ..., t+k-t'-1$.

To complete the proof, it suffices to establish that $p_{t'}^m(x_{t'})$ is weakly increasing in $x_{t'}$ for all $m = 1, ..., t+k-t'$, since then we may conclude by induction that $p_t^k(x_t)$ is weakly increasing in x_t. (The argument applies for all $k \geq 1$.)

First, Theorem 3(i) implies that $p_{t'}^1(x_{t'})$ is weakly increasing in $x_{t'}$. For all $m > 0$,

$$p_{t'}^m(x_{t'}) = p_{t'}^1(x_{t'}) E\left[p_{t'+1}^{m-1}(x_{t'}, X_{t'+1}) \middle| x_{t'}\right] \tag{22}$$

(The partnership survives for m periods iff it survives for $m-1$ periods after first surviving for one period.) The base step showed that the first term of (22) is weakly increasing in $x_{t'}$. It suffices now to show the same of the expectation term

$$E\left[p_{t'+1}^{m-1}(x_{t'}, X_{t'+1}) \middle| x_{t'}\right] = \int_0^1 \Pr\left(p_{t'+1}^{m-1}(x_{t'}, X_{t'+1}) > p \middle| x_{t'}\right) \mathrm{d}p \tag{23}$$

By the induction hypothesis, each set $\{x_{t'+1} \in \mathcal{X}_{t'+1} : p_{t'+1}^{m-1}(x_{t'+1}; x_{t'}) > p\}$ is both an increasing subset of $\mathcal{X}_{t'+1}$ and weakly increasing in $x_{t'}$ (relative to set inclusion). By Assumption 4, we conclude that each of the probability-terms in (23) is weakly increasing in $x_{t'}$. This completes the proof. □