ML-1
7/22/11

Solution Methods for DSGE Models and Applications using Linearization

Lawrence J. Christiano

Overall Outline

- Perturbation and Projection Methods for DSGE Models: an Overview

- Simple New Keynesian model
  - Formulation and log-linear solution.
  - Ramsey-optimal policy.
  - Using Dynare to solve the model by log-linearization:
    - Taylor principle, implications of working capital, News shocks, monetary policy with the long rate.

- Financial Frictions as in BGG
  - Risk shocks and the CKM critique of intertemporal shocks.
  - Dynare exercise.

- Ramsey Optimal Policy, Time Consistency, Timeless Perspective.

Perturbation and Projection Methods for Solving DSGE Models

Lawrence J. Christiano

Outline

- A Simple Example to Illustrate the basic ideas.
  - Functional form characterization of model solution.
  - Use of Projections and Perturbations.

- Neoclassical model.
  - Projection methods
  - Perturbation methods
    - Make sense of the proposition, ‘to a first order approximation, can replace equilibrium conditions with linear expansion about nonstochastic steady state and solve the resulting system using certainty equivalence’

Simple Example

- Suppose that *x* is some exogenous variable and that the following equation implicitly defines *y*:

$$h(x,y) = 0, \text{ for all } x \in X$$

- Let the solution be defined by the 'policy rule', *g*:

$$y = g(x)$$

'Error function'

- satisfying

$$R(x;g) \equiv h(x,g(x)) = 0$$

- for all $x \in X$

The Need to Approximate

- Finding the policy rule, *g*, is a big problem outside special cases
  - ‘Infinite number of unknowns (i.e., one value of *g* for each possible *x*) in an infinite number of equations (i.e., one equation for each possible *x*).’
- Two approaches:
  - projection and perturbation

Projection

- Find a parametric function, $\hat{g}(x;\gamma)$, where γ is a vector of parameters chosen so that it imitates the property of the exact solution, i.e., $R(x;g) = 0$ for all $x \in X$, as well as possible.
- Choose values for γ so that

$$\hat{R}(x;\gamma) = h(x, \hat{g}(x;\gamma))$$

- is close to zero for $x \in X$.

- The method is defined by how 'close to zero' is defined and by the parametric function, $\hat{g}(x;\gamma)$, that is used.

Projection, continued

- Spectral and finite element approximations
  - **Spectral functions**: functions, $\hat{g}(x;\gamma)$, in which each parameter in γ influences $\hat{g}(x;\gamma)$ for all $x \in X$ example:

$$\hat{g}(x;\gamma) = \sum_{i=0}^{n} \gamma_i H_i(x),\ \gamma = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}$$

$$H_i(x) = x^i \text{ ~ordinary polynominal (not computationaly efficient)}$$

$$H_i(x) = T_i(\varphi(x)),$$

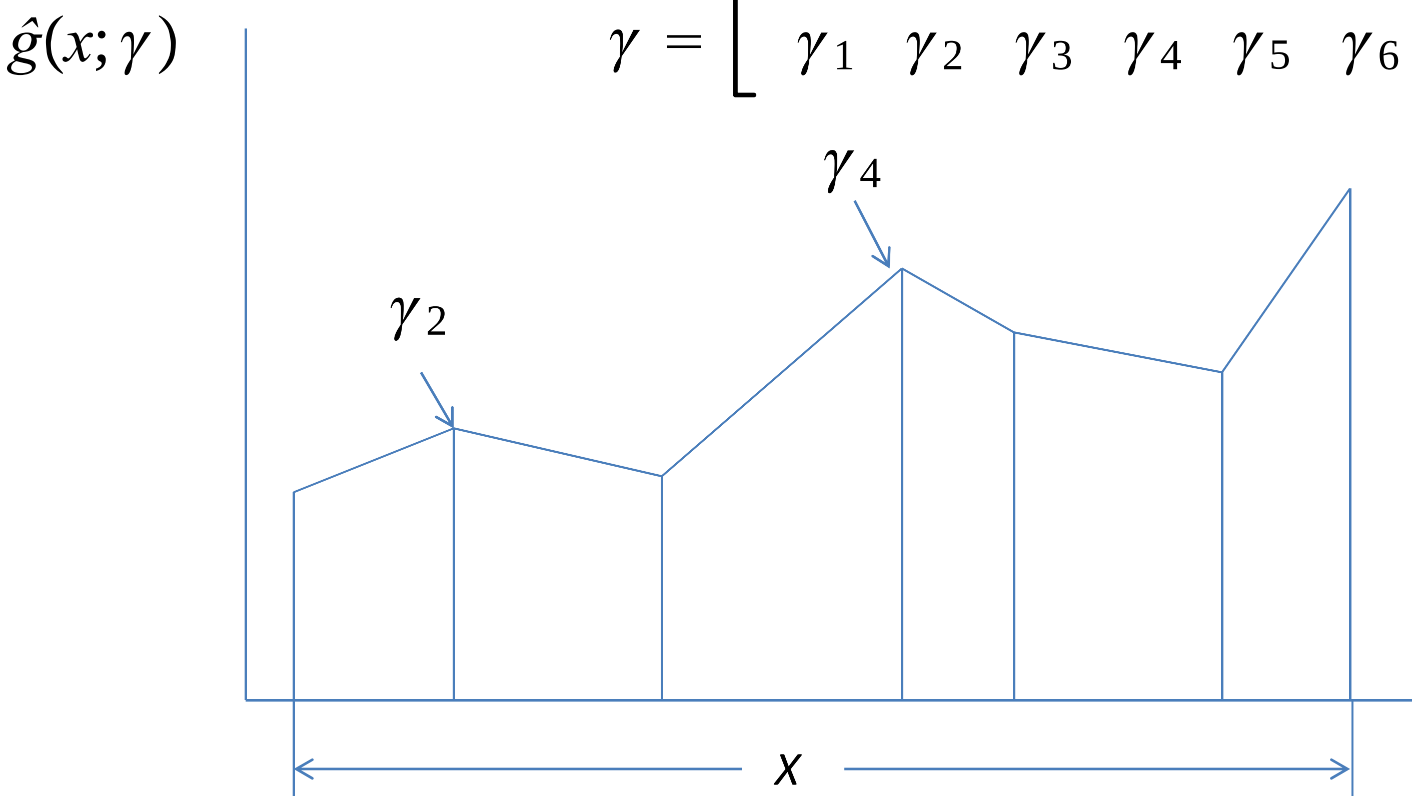
$$T_i(z) : [-1,1] \to [-1,1],\ i^{th} \text{ order Chebyshev polynomial}$$

$$\varphi : X \to [-1,1]$$

Projection, continued

– Finite element approximations: functions, $\hat{g}(x;\gamma)$, in which each parameter in γ influences $\hat{g}(x;\gamma)$ over only a subinterval of $x \in X$

$$\gamma = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 \end{bmatrix}$$



Projection, continued

- 'Close to zero': collocation and Galerkin
- **Collocation**, for *n* values of $x: \; x_1, x_2, \ldots, x_n \in X$ choose *n* elements of $\gamma = \begin{bmatrix} \gamma_1 & \cdots & \gamma_n \end{bmatrix}$ so that

$$\hat{R}(x_i;\gamma) = h(x_i, \hat{g}(x_i;\gamma)) = 0, \; i = 1, \ldots, n$$

  – how you choose the grid of *x*'s matters…

- **Galerkin**, for *m>n* values of $x: \; x_1, x_2, \ldots, x_m \in X$ choose the n elements of $\gamma = \begin{bmatrix} \gamma_1 & \cdots & \gamma_n \end{bmatrix}$

$$\sum_{j=1}^{m} w_j^i h(x_j, \hat{g}(x_j;\gamma)) = 0, \; i = 1, \ldots, n$$

Perturbation

- Projection uses the 'global' behavior of the functional equation to approximate solution.
  - Problem: requires finding zeros of non-linear equations. Iterative methods for doing this are a pain.
  - Advantage: can easily adapt to situations the policy rule is not continuous or simply non-differentiable (e.g., occasionally binding zero lower bound on interest rate).

- Perturbation method uses local properties of functional equation and Implicit Function/Taylor's theorem to approximate solution.
  - Advantage: can implement it using non-iterative methods.
  - Possible disadvantages:
    - may require enormously high derivatives to achieve a decent global approximation.
    - Does not work when there are important non-differentiabilities (e.g., occasionally binding zero lower bound on interest rate).

Perturbation, cnt'd

- Suppose there is a point, $x^* \in X$, where we know the value taken on by the function, *g*, that we wish to approximate:

$$g(x^*) = g^*, \text{ some } x^*$$

- Use the implicit function theorem to approximate *g* in a neighborhood of x^*
- Note:

$$R(x;g) = 0 \text{ for all } x \in X$$

$$\rightarrow$$

$$R^{(j)}(x;g) \equiv \frac{d^j}{dx^j} R(x;g) = 0 \text{ for all } j, \text{ all } x \in X.$$

Perturbation, cnt'd

- Differentiate *R* with respect to x and evaluate the result at x^* :

$$R^{(1)}(x^*) = \frac{d}{dx}h(x,g(x))|_{x=x^*} = h_1(x^*,g^*) + h_2(x^*,g^*)g'(x^*) = 0$$

$$\rightarrow g'(x^*) = -\frac{h_1(x^*,g^*)}{h_2(x^*,g^*)}$$

- Do it again!

$$R^{(2)}(x^*) = \frac{d^2}{dx^2}h(x,g(x))|_{x=x^*} = h_{11}(x^*,g^*) + 2h_{12}(x^*,g^*)g'(x^*)$$

$$h_{22}(x^*,g^*)[g'(x^*)]^2 + h_2(x^*,g^*)g''(x^*).$$

$\rightarrow$ Solve this linearly for $g''(x^*)$.

Perturbation, cnt'd

- Preceding calculations deliver (assuming enough differentiability, appropriate invertibility, a high tolerance for painful notation!), recursively:

$$g'(x^*), g''(x^*), \ldots, g^{(n)}(x^*)$$

- Then, have the following Taylor's series approximation:

$$g(x) \approx \hat{g}(x)$$

$$\hat{g}(x) = g^* + g'(x^*) \times (x - x^*)$$

$$+ \frac{1}{2} g''(x^*) \times (x - x^*)^2 + \ldots + \frac{1}{n!} g^{(n)}(x^*) \times (x - x^*)^n$$

Perturbation, cnt'd

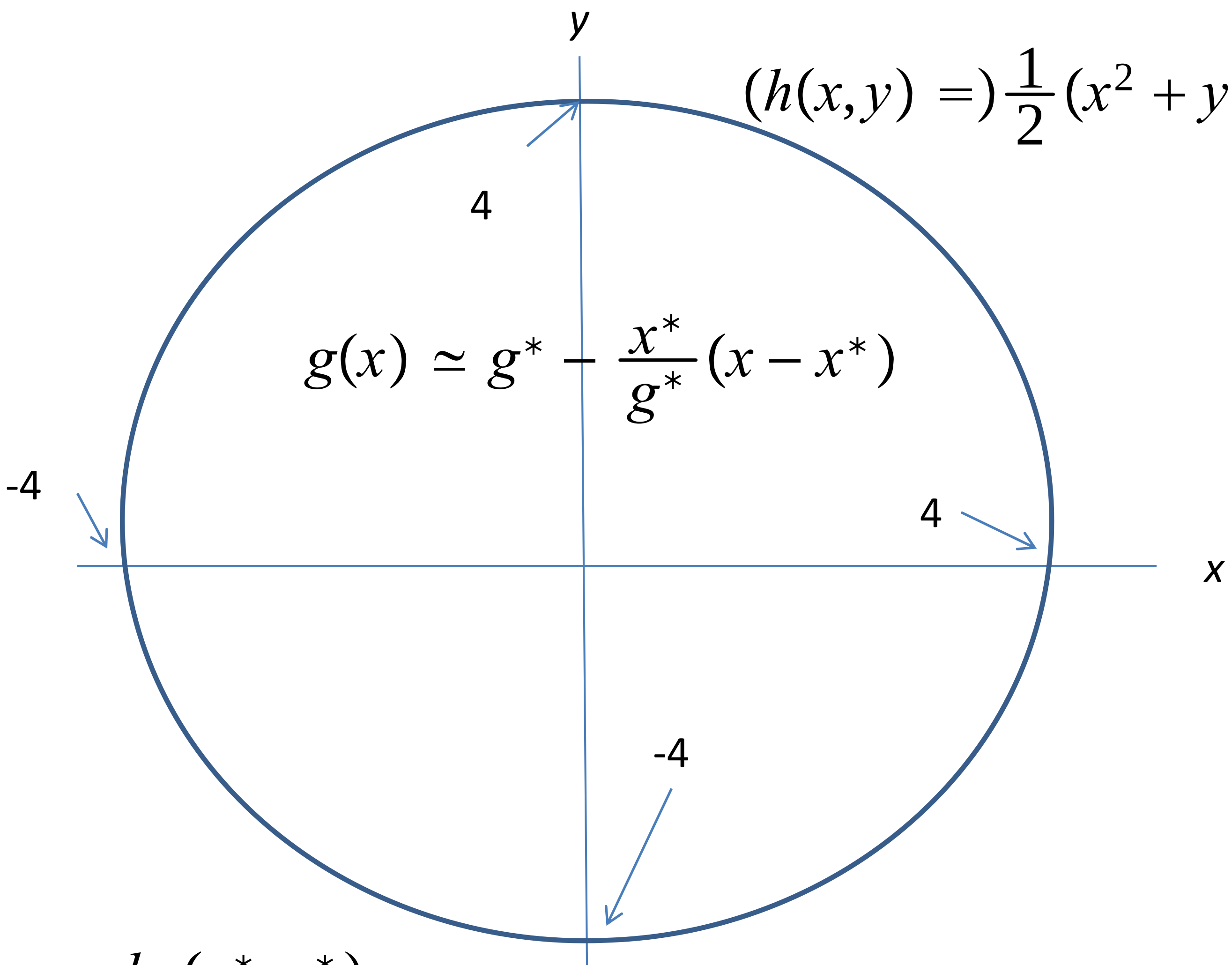
- Check….
- Study the graph of

$$R(x;\hat{g})$$

  – over $x \in X$ to verify that it is everywhere close to zero (or, at least in the region of interest).

Example of Implicit Function Theorem

$$(h(x,y) =) \frac{1}{2}(x^2 + y^2) - 8 = 0$$



$$g(x) \simeq g^* - \frac{x^*}{g^*}(x - x^*)$$

$$g'(x^*) = -\frac{h_1(x^*, g^*)}{h_2(x^*, g^*)} = -\frac{x^*}{g^*} \text{ } (h_2 \text{ had better not be zero!})$$

Neoclassical Growth Model

- Objective:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),\ u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$

- Constraints:

$$c_t + \exp(k_{t+1}) \le f(k_t, a_t),\ t = 0, 1, 2, \ldots.$$

$$a_t = \rho a_{t-1} + \varepsilon_t.$$

$$f(k_t, a_t) = \exp(\alpha k_t)\exp(a_t) + (1-\delta)\exp(k_t).$$

Efficiency Condition

$$E_t[u'\left(\overbrace{f(k_t, a_t) - \exp(k_{t+1})}^{c_{t+1}}\right)$$

$$-\beta u'\left(\overbrace{f(k_{t+1}, \rho a_t + \varepsilon_{t+1}) - \exp(k_{t+2})}^{c_{t+1}}\right) \overbrace{f_K(k_{t+1}, \rho a_t + \varepsilon_{t+1})}^{\text{period } t+1 \text{ marginal product of capital}} \quad] = 0.$$

- Here,

k_t, a_t ~given numbers

$\varepsilon_{t+1} \sim iid$, mean zero variance V_ε

time t choice variable, k_{t+1}

- Convenient to suppose the model is the limit of a sequence of models, $\sigma \to 1$, indexed by σ

$$\varepsilon_{t+1} \sim \sigma^2 V_\varepsilon,\ \sigma = 1.$$

Solution

- A policy rule,

$$k_{t+1} = g(k_t, a_t, \sigma).$$

- With the property:

$$R(k_t, a_t, \sigma; g) \equiv E_t\{u'\left(\overbrace{f(k_t, a_t) - \exp[g(k_t, a_t, \sigma)]}^{c_t}\right)$$

$$-\beta u'\left(\overbrace{f\left(\overbrace{g(k_t, a_t, \sigma)}^{k_{t+1}}, \overbrace{\rho a_t + \sigma\varepsilon_{t+1}}^{a_{t+1}}\right) - \exp\left[g\left(\overbrace{g(k_t, a_t, \sigma)}^{k_{t+1}}, \overbrace{\rho a_t + \sigma\varepsilon_{t+1}}^{a_{t+1}}, \sigma\right)\right]}^{c_{t+1}}\right)$$

$$\times f_K\left(\overbrace{g(k_t, a_t, \sigma)}^{k_{t+1}}, \overbrace{\rho a_t + \sigma\varepsilon_{t+1}}^{a_{t+1}}\right)\} = 0,$$

- for all a_t, k_t and $\sigma = 1.$

Projection Methods

- Let

$$\hat{g}(k_t, a_t, \sigma; \gamma)$$

  - be a function with finite parameters (could be either spectral or finite element, as before).

- Choose parameters, γ, to make

$$R(k_t, a_t, \sigma; \hat{g})$$

  - as close to zero as possible, over a range of values of the state.
  - use Galerkin or Collocation.

Occasionally Binding Constraints

- Suppose we add the non-negativity constraint on investment:

$$\exp(g(k_t, a_t, \sigma)) - (1-\delta)\exp(k_t) \geq 0$$

- Express problem in Lagrangian form and optimum is characterized in terms of equality conditions with a multiplier and with a complementary slackness condition associated with the constraint.

- Conceptually straightforward to apply preceding method. For details, see Christiano-Fisher, 'Algorithms for Solving Dynamic Models with Occasionally Binding Constraints', 2000, *Journal of Economic Dynamics and Control.*
  – This paper describes alternative strategies, based on parameterizing the expectation function, that may be easier, when constraints are occasionally binding constraints.

Perturbation Approach

- Straightforward application of the perturbation approach, as in the simple example, requires knowing the value taken on by the policy rule at a point.

- The overwhelming majority of models used in macro do have this property.

  – In these models, can compute non-stochastic steady state without any knowledge of the policy rule, *g*.
  – Non-stochastic steady state is k^* such that

$$k^* = g\left(k^*, \overbrace{0}^{a=0 \text{ (nonstochastic steady state in no uncertainty case)}}, \overbrace{0}^{\sigma=0 \text{ (no uncertainty)}} \right)$$

  – and

$$k^* = \log\left\{ \left[\frac{\alpha\beta}{1-(1-\delta)\beta} \right]^{\frac{1}{1-\alpha}} \right\}.$$

Perturbation

- Error function:

$$R(k_t, a_t, \sigma; g) \equiv E_t\{u'\left(\overbrace{f(k_t, a_t) - \exp[g(k_t, a_t, \sigma)]}^{c_t}\right)$$

$$- \beta u'\left[\overbrace{f(g(k_t, a_t, \sigma), \rho a_t + \sigma\varepsilon_{t+1}) - \exp[g(g(k_t, a_t, \sigma), \rho a_t + \sigma\varepsilon_{t+1}, \sigma)]}^{c_{t+1}}\right]$$

$$\times f_K(g(k_t, a_t, \sigma), \rho a_t + \sigma\varepsilon_{t+1})\} = 0,$$

  – for all values of k_t, a_t, σ.

- So, all order derivatives of *R* with respect to its arguments are zero (assuming they exist!).

Four (Easy to Show) Results About Perturbations

- Taylor series expansion of policy rule:

$$g(k_t, a_t, \sigma) \simeq \overbrace{k + g_k(k_t - k) + g_a a_t + g_\sigma \sigma}^{\text{linear component of policy rule}}$$

$$\overbrace{+\frac{1}{2}[g_{kk}(k_t - k)^2 + g_{aa}a_t^2 + g_{\sigma\sigma}\sigma^2] + g_{ka}(k_t - k)a_t + g_{k\sigma}(k_t - k)\sigma + g_{a\sigma}a_t\sigma + \ldots}^{\text{second and higher order terms}}$$

- $g_\sigma = 0$: to a first order approximation, 'certainty equivalence'
- All terms found by solving linear equations, except coefficient on past endogenous variable, g_k ,which requires solving for eigenvalues
- To second order approximation: slope terms certainty equivalent –

  $$g_{k\sigma} = g_{a\sigma} = 0$$
- Quadratic, higher order terms computed recursively.

First Order Perturbation

- Working out the following derivatives and evaluating at $k_t = k^*, a_t = \sigma = 0$

$$R_k(k_t, a_t, \sigma; g) = R_a(k_t, a_t, \sigma; g) = R_\sigma(k_t, a_t, \sigma; g) = 0$$

- Implies:

‘problematic term’

Source of certainty equivalence
In linear approximation

$$R_k = u''(f_k - e^g g_k) - \beta u' f_{Kk} g_k - \beta u''(f_k g_k - e^g g_k^2) f_K = 0$$

$$R_a = u''(f_a - e^g g_a) - \beta u'[f_{Kk} g_a + f_{Ka}\rho] - \beta u''(f_k g_a + f_a \rho - e^g[g_k g_a + g_a \rho]) f_K = 0$$

$$R_\sigma = -[u' e^g + \beta u''(f_k - e^g g_k) f_K] g_\sigma = 0$$

Technical notes for following slide

$$
\begin{aligned}
u''(f_k - e^g g_k) - \beta u' f_{Kk} g_k - \beta u''(f_k g_k - e^g g_k^2) f_K &= 0 \\
\frac{1}{\beta}(f_k - e^g g_k) - u' \frac{f_{Kk}}{u''} g_k - (f_k g_k - e^g g_k^2) f_K &= 0 \\
\frac{1}{\beta} f_k - \left[\frac{1}{\beta} e^g + u' \frac{f_{Kk}}{u''} + f_k f_K \right] g_k + e^g g_k^2 f_K &= 0 \\
\frac{1}{\beta} \frac{f_k}{e^g f_K} - \left[\frac{1}{\beta f_K} + \frac{u'}{u''} \frac{f_{Kk}}{e^g f_K} + \frac{f_k}{e^g} \right] g_k + g_k^2 &= 0 \\
\frac{1}{\beta} - \left[1 + \frac{1}{\beta} + \frac{u'}{u''} \frac{f_{Kk}}{e^g f_K} \right] g_k + g_k^2 &= 0
\end{aligned}
$$

- Simplify this further using:

$$
\begin{aligned}
f_K &= \alpha K^{\alpha-1} \exp(a) + (1-\delta),\ K \equiv \exp(k) \\
&= \alpha \exp[(\alpha-1)k + a] + (1-\delta) \\
f_k &= \alpha \exp[\alpha k + a] + (1-\delta) \exp(k) = f_K e^g \\
f_{Kk} &= \alpha(\alpha-1) \exp[(\alpha-1)k + a] \\
f_{KK} &= \alpha(\alpha-1) K^{\alpha-2} \exp(a) = \alpha(\alpha-1) \exp[(\alpha-2)k + a] = f_{Kk} e^{-g}
\end{aligned}
$$

- to obtain polynomial on next slide.

First Order, cont'd

- Rewriting $R_k = 0$ term:

$$\frac{1}{\beta} - \left[1 + \frac{1}{\beta} + \frac{u'}{u''} \frac{f_{KK}}{f_K} \right] g_k + g_k^2 = 0$$

- There are two solutions, $0 < g_k < 1,\ g_k > \frac{1}{\beta}$
  - Theory (see Stokey-Lucas) tells us to pick the smaller one.
  - In general, could be more than one eigenvalue less than unity: multiple solutions.
- Conditional on solution to g_k, g_a solved for linearly using $R_a = 0$ equation.
- These results all generalize to multidimensional case

Numerical Example

- Parameters taken from Prescott (1986):

$$\gamma = 2\ (20),\ \alpha = 0.36,\ \delta = 0.02,\ \rho = 0.95,\ V_e = 0.01^2$$

- Second order approximation:

$$\hat{g}(k_t, a_{t-1}, \varepsilon_t, \sigma) = \overbrace{k^*}^{3.88} + \overbrace{g_k}^{0.98\ (0.996)} (k_t - k^*) + \overbrace{g_a}^{0.06\ (0.07)} a_t + \overbrace{g_\sigma}^{0} \sigma$$

$$+ \frac{1}{2}[\ \overbrace{g_{kk}}^{0.014\ (0.00017)} (k_t - k)^2 + \overbrace{g_{aa}}^{0.067\ (0.079)} a_t^2 + \overbrace{g_{\sigma\sigma}}^{0.000024\ (0.00068)} \overbrace{\sigma^2}^{1}\ ]$$

$$+ \overbrace{g_{ka}}^{-0.035\ (-0.028)} (k_t - k) a_t + \overbrace{g_{k\sigma}}^{0} (k_t - k)\sigma + \overbrace{g_{a\sigma}}^{0} a_t \sigma$$

Conclusion

- For modest US-sized fluctuations and for aggregate quantities, it is reasonable to work with first order perturbations.

- First order perturbation: linearize (or, log-linearize) equilibrium conditions around non-stochastic steady state and solve the resulting system.
  - This approach assumes ‘certainty equivalence’. Ok, as a first order approximation.

List of endogenous variables determined at *t*

Solution by Linearization

- (log) Linearized Equilibrium Conditions:

$$E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0$$

- Posit Linear Solution:

$$z_t = A z_{t-1} + B s_t$$

$$s_t - P s_{t-1} - \epsilon_t = 0.$$

Exogenous shocks

- To satisfy equil conditions, *A* and *B* must:

$$\alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0, \;\; F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0$$

- If there is exactly one *A* with eigenvalues less than unity in absolute value, that's the solution. Otherwise, multiple solutions.

- Conditional on *A*, solve linear system for *B*.